Problem Set 7

Problems 5.2, 5.4, 5.8 in Sprott

Then, some practice with the convolution theorem:

(Note: You will need only the four transform pairs at the top of our "cheatsheet".)

1. If you have a voltage signal *V(t)* and would like to know its frequency spectrum *V(f),* you will probably want to use a computer to approximate the Fourier transform integral. But the integral requires a continuous function in time, and you can only digitize *V(t)* at some interval Δt and put this discrete set of numbers into the computer and do the Fourier transform on them.

You could imagine generating the discrete set of points by taking the product of *V(t)* with a "picket fence" of delta functions spaced Δt apart. Use the convolution theorem to prove the Nyquist sampling theorem: If *V(f)* is identically zero for all $f > f_{\mathrm{NYQUIST}}$, where $f_{\mathrm{NYQUIST}} \equiv 1/2\Delta t$, then the F.T. of the sampled waveform will be identical to *V(f)* for $f < f_{\mathrm{NYQUIST}}$.

Make a sketch of the functions you used with the convolution theorem to show how this works. Don't forget the negative frequencies.

2. Another problem with your computed approximation for V(f) is that the Fourier integral goes from $-\infty < t < \infty$, and you probably don't want to take data for that long. Suppose you take data from $t = -T/2$ to $t = +T/2$. Use the convolution theorem to describe the effect on an arbitrary spectrum. Sketch both the true *V(f)* and the *V(f)* obtained from the computer for $V(t) = \cos(2\pi 9t) + \cos(2\pi 11t)$. What is the minimum sampling rate required to avoid aliasing?

3. Show that the Heisenberg uncertainty principle, $\Delta p \Delta x \geq \frac{\hbar}{2}$, is an exact equality if the probability distribution for the position, x, is gaussian. Δp and Δx are the r.m.s. uncertainties in position and momentum. Note that the r.m.s. deviation of a gaussian $e^{\frac{\Delta x^2}{2\sigma^2}}$ is σ, and the square of a gaussian is another gaussian with r.m.s. deviation smaller by $1/\sqrt{2}$.

For non-physicists: The momentum $p = \frac{h}{\lambda} \equiv hw$. ($w \equiv 1/\lambda$ is called the "wavenumber") The probability distribution in position is $|\Psi(x)|^2$ and the probability distribution in w is $|\Psi(w)|^2$, where $\Psi(x)$ and $\Psi(w)$ are a Fourier transform pair. For a pure momentum state (perfectly defined momentum $p = hw_0$), $\Psi(x) = \Psi_0 e^{i2\pi w_0 x}$. To localize this with a probability distribution in x around $x = x_0$, we can multiply this $\Psi(x)$ by a gaussian in x to make a "wave packet": $\Psi(x) = \Psi_0 e^{i2\pi w_0 x} e^{(x-x_0)^2/2\sigma^2}$. (note that $\hbar \equiv h/2\pi$)