# Experiment 6
# Fourier Analysis of a Square Wave

## 1 Motivation

Information processed by electronics is rarely single-frequency sinusoidal waves. Measurements of real-world systems need to be capable of capturing complex time dependence. Digital circuity operates with rapid transitions from low-to-high potential in non-periodic trains of square pulses. Both non-sinusoidal periodic and non-periodic waveforms can be treated as a superposition of sinusoidal waves, i.e., Fourier series and Fourier integrals. This experiment will expose the Fourier harmonics of a periodic square wave to illustrate a non-sinusoidal wave's harmonic content.

## 2 Background

Any periodic wave can be viewed as a Fourier series (superposition) of sinusoidal waves with frequencies $\omega_n = n\omega_0$, where $\omega_0 = 2\pi f_0$ is the fundamental frequency and $n$ is the harmonic number. The Fourier series for a periodic square wave is

$$\begin{aligned} v(t) &= \sum_{n=1}^{\infty} v_n \sin(n\omega_0 t) \\ v_n &= \frac{4V_0}{n\pi}, \qquad n = 1, 3, 5, 7, 9... \\ v_n &= 0, \qquad n = 2, 4, 6, 8... \end{aligned} \tag{1}$$

When the arbitrary function generator is set to produce a "square wave", the value of the frequency you enter is $f_0$, and the generator synthesizes a waveform that comes close to replicating a perfect square wave. In this experiment you will use a resonant $R$-$L$-$C$ filter to isolate and measure the amplitudes, $|v_n|$, of the first few harmonics of a square wave. Beware! There is a duplicate use of "$f_0$" for both the periodic wave's fundamental frequency and an $R$-$L$-$C$ filter's resonant frequency. In this experiment we will use $f_0$ to mean the square wave's fundamental frequency and refer explicitly to the filter's resonant frequency as $1/2\pi\sqrt{LC}$.

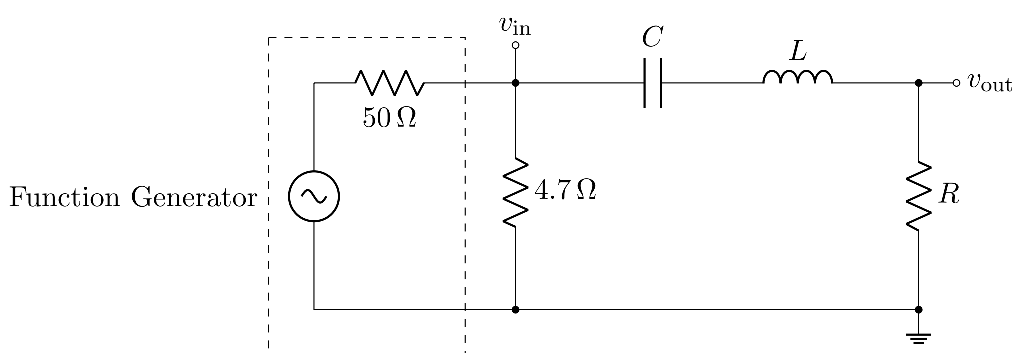


Figure 1: A low-impedance resonant filter. The dashed-line box identifies the function generator. Internal to the generator there is explicitly a 50 Ω resistor in series with its output.

The resonant filter circuit you will use is shown in Fig. 1. You will tune the filter by adjusting the capacitor, $C$, so that the filter is resonant for one particular harmonic at a time. This works best if the resonance has a large quality factor, $Q$, and the pass-band is very narrow. The circuit in Fig. 1 is almost identical to the resonant $R$-$L$-$C$ filter in Lab 5, but there is the addition of a $4.7\,\Omega$ resistor across the output of the function generator. This lowers the output impedance of the source and helps increase $Q$.

## 3 Equipment

For this lab, you will use:

- One Tektronix MSO 2014B Digital Storage Oscilloscope
- One AFG2021 Arbitrary Function Generator
- One ELC variable resistance box
- One ELC variable capacitance box
- One inductor coil* mounted on a circuit board
- One $4.7\,\Omega$, 1 W carbon-compositon resistor (from parts drawer cabinet)

* If you select the numbered inductor board you used for the AC Wheatstone Bridge then you will know the value of $L$ to within 2%, but this is not crucial.

## 4 Procedure

You will measure the amplitude of the first nine Fourier harmonics of a $f_0 = 3\,\text{kHz}$ square wave. Your experimental data should be recorded in a table with the following quantities:

- harmonic number, $n$
- frequency of the harmonic, $f_n$
- estimated capacitance, $C$, needed to tune the resonance for each harmonic (step 4)
- actual capacitance, $C$, needed to tune the resonance for each harmonic (step 5)
- peak-to-peak $|v_{\text{in}}|$ and $|v_{\text{out}}| = |v_R|$ for each harmonic (steps 3-5)
- ratio $|v_{\text{out}}|/|v_{\text{in}}|$ for each harmonic
- sinusoidal rms $|v_{\text{in,s}}|$ and $|v_{\text{out,s}}|$ at resonance for each $f_n$ (step 6)
- sinusoidal filter attenuation, $A = |v_{\text{out,s}}/v_{\text{in,s}}|$ for each $f_n$ (step 6)
- corrected ratio $(|v_{\text{out}}|/|v_{\text{in}}|)/A$ for each harmonic (step 7)
- theoretical ratio of Fourier amplitudes and percentage errors (step 8)

Set up the circuit shown in Fig. 1. Use the oscilloscope for all of your voltage measurements. (Do not use a scope probe.) Use a BNC "Y" or "T" adapter to make a parallel connection from the function generator to the scope to measure $v_{\text{in}}$. Use a BNC-to-banana jack adapter to connect the generator to the circuit. Use the screw terminals of the BNC-to-banana jack adapter to install the $4.7\,\Omega$ resistor. (Ask your lab instructor how to do this.) Set $R = 40\,\Omega$ using the "×10" decade of the variable resistor box. The inductor mounted on the circuit board is $L = 20\,\text{mH} \pm 15\%$. The capacitor, $C$, will be varied in the steps below, but set its initial value such that the resonant frequency $1/2\pi\sqrt{LC} = 3\,\text{kHz}$ to match the fundamental frequency, $f_0$.

1. Start by setting the function generator to produce a 3 kHz *sine wave* with an amplitude of 10 V peak-to-peak. The voltage divider formed by the internal $50\,\Omega$ and the external $4.7\,\Omega$ will attenuate the function generator's output, so expect $v_{\text{in}} \approx 1$ V peak-to-peak. Adjust $C$

to cause the filter to resonate at 3 kHz. The most sensitive detection of the resonance is zero phase shift between $v_R$ and $v_{\text{in}}$. How close is the required $C$ to your estimate above? Is is consistent with the uncertainty in $L$?

2. Now set the function generator to make a 3 kHz *square wave*. Most likely you will no longer see stable waveforms on the scope. This is due to high-frequency transients at the edges of the square wave confusing the scope trigger. To create a stable display, use a BNC coaxial cable to connect the trigger output of the function generator to a third channel in the scope and use it as the scope's trigger source. The generator's trigger signal is an offset square wave with frequency identical to the output voltage. It is intended to be used for timing in situations like this. It is not required to display the trigger source, so you can turn it off or move it to the top of the scope's display so it is out of the way.
3. When the display is stable, you should see that the relative phase between $v_R$ and $v_{\text{in}}$ is still close to zero (in resonance). You will also see that the "square" wave is not so square. This is due to frequency-dependent loading by the circuit. To see the undistorted square wave source, break the circuit at one point by disconnecting one end of one patch cable. The $v_{\text{in}}$ waveform should now be close to an ideal square wave. Reconnect the patch cable and measure the *peak-to-peak* amplitudes of $|v_{\text{in}}|$ and $|v_{\text{out}}| = |v_R|$ using the scope's cursors. The waveforms you will encounter in this experiment will have various distortions, so the peak amplitude measured using the cursors (guided by eye) will provide a more consistent measure than rms. Place the cursor at a level that estimates the average value of the peak(s). This is your data for the fundamental harmonic ($n = 1$). Be sure to record the value of $C$ required to tune the resonance.
4. Now measure the peak-to-peak amplitudes for the odd harmonics, $n = 3$, 5, 7 and 9. To do this, decrease $C$ so that the circuit resonates at $f_n = nf_0$. It will help to take a minute and estimate in advance the value of $C$ required to tune the filter's resonance for each harmonic, $f_n = n/2\pi\sqrt{LC_0}$, where $C_0$ is the capacitance required for $f_0$ (determined in step 3). You can do this now for all $n = 2$ to 9 and enter your initial estimates for $C$ in your table. (Be sure to use your *measured* $C_0$.) Use these estimates for $C$ as the starting points for each harmonic. For larger $n$, the capacitor box's step changes in $C$ become relatively coarse. To fine tune the resonance further, make small adjustments in $f_0$ (the square wave fundamental frequency) but no more than a few percent. Set the frequency back to 3 kHz when you proceed to next harmonic. As before, the relative phase (measured and visual) is the most sensitive indicator of the resonance condition. You will notice that $v_{\text{out}}$ is increasingly distorted, with "kicks" at the step transitions of the square wave. These are transients that decay exponentially. Place the cursors to estimate the average peak amplitude. Don't forget to record your actual values for $C$. Make sketches of $v_{\text{out}}(t)$ for $n = 3$ and 5.
5. Now measure the amplitudes of the even harmonics, $n = 2$, 4, 6, 8. Set the frequency to 3 kHz (if it was changed in the fine tuning process). For each harmonic, set the capacitor to your estimated value and follow the procedure as in steps 3 and 4. The $v_{\text{out}}$ waveforms will be quite distorted (not sinusoidal), so you cannot easily see the phase shift. Adjust $C$ in small amounts to observe the minimum in $v_{\text{out}}$. Try varying the square wave frequency up and down $\pm 10\%$ (relative to $f_0$) and observe what happens to $v_{\text{out}}$. Record the peak-to-peak $v_{\text{in}}$, minimum $v_{\text{out}}$, and the actual $C$ you used for each $n$. Make sketches of $v_{\text{out}}(t)$ for $n = 2$ and 4.
6. If you look carefully at your measurements, you will notice that the ratio $|v_{\text{out}}|/|v_{\text{in}}|$ for the odd harmonics does not scale as $1/n$. This is because the filter's gain is frequency-dependent. Ideally, the filter gain is $A = 1$ at resonance for any frequency, but the inductor's finite resistance forms a voltage divider with the circuit's $R$, so the gain $A < 1$. To measure $A$, set

the function generator to make a sine wave (as in step 1). Now scan frequency, setting the function generator to $f = f_n$ for each harmonic. For each $f_n$, set $C$ to the value you used to fine tune the resonance. In each case you should fine tune the resonance by making small changes in the frequency relative to $f_n$ while monitoring the phase, since your goal is a precise measurement of the filter's gain at resonance near $f_n$. Record either the peak-to-peak or rms amplitudes of $v_{\mathrm{in,s}}$ and $v_{\mathrm{out,s}}$, where the subscript "$s$" signifies sine wave. For sine waves, you can reliably use the scope's rms measurement. Calculate and record $A = |v_{\mathrm{out,s}}|/|v_{\mathrm{in,s}}|$ for each harmonic. Make a plot of $A$ versus $f_n$.

7. Now calculate $(|v_{\mathrm{out}}|/|v_{\mathrm{in}}|)/A$ for each harmonic, which corrects for the filter's frequency-dependent gain. Enter the gain-corrected ratios in your table.
8. Compare your measurements with the ideal Fourier series for a symmetric square wave, Eqs. 1. For the odd harmonics, calculate and record the percentage error. Errors of 5-10% are reasonable for this experiment. Make a plot that summarizes your data and comparison with the ideal Fourier series.
9. Can you explain why the waveforms for the even harmonics look the way they do? What would happen if you used a resonant filter with higher $Q$? (Your circuit has $Q \approx 10$.)