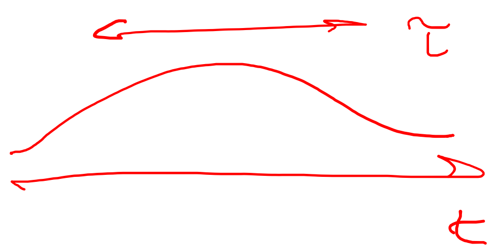
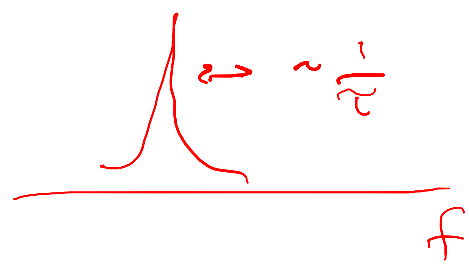
# Fourier

$$\tilde{v}(f) = \int_{-\infty}^{\infty} v(t)\, e^{-j2\pi f t}\, dt$$

$$v(t) = \int_{-\infty}^{\infty} \tilde{v}(f)\, e^{+j2\pi f t}\, df$$



# CONVOLUTION THEOREM

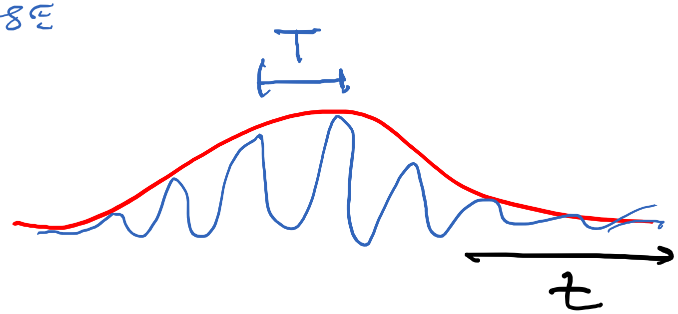
DEFINITION: $f * g = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$

$\rightarrow \widetilde{f * g} = \tilde{f} \cdot \tilde{g}$

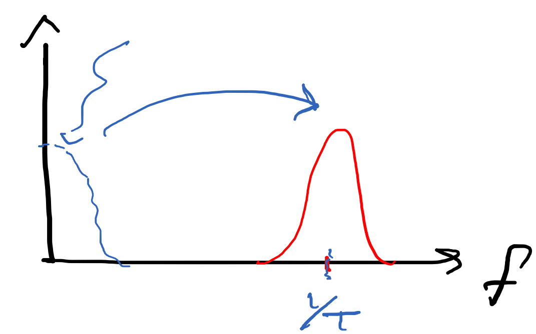
$\rightarrow \widetilde{f \cdot g} = \tilde{f} * \tilde{g}$

. . . . . . .

IMPORTANT CASE

$\rightarrow$ 


ENVELOPE ALONE



Proof.

$$\widetilde{f*g} = \int_t \int_\tau f(t-\tau)\, g(\tau)\, e^{-2\pi j f t}\, d\tau\, dt$$

$$x = t - \tau$$

$$= \int_x \int_\tau f(x)\, g(\tau)\, e^{-2\pi j f x}\, e^{-2\pi j f \tau}\, dx\, d\tau$$

$$= \tilde{f} \cdot \tilde{g}$$

$$\widetilde{f \cdot g} = \int f(t)\, g(t)\, e^{-2\pi j f t}\, dt$$

$$\rightarrow g(t) = \int \tilde{g}(f')\, e^{+2\pi j f' t}\, df'$$

$$= \int_{f'} \int_t \tilde{g}(f')\, \underbrace{f(t)\, e^{2\pi j (f'-f) t}}\, dt\, df'$$

$$= \int_{f'} \tilde{g}(f')\, \tilde{f}(f - f')\, df' = \tilde{f} * \tilde{g}$$

# PARSEVAL

$$\int_{-\infty}^{\infty} |v(t)|^2 \, dt = \int_{-\infty}^{\infty} |\tilde{v}(f)|^2 \, df$$

CONSERVATION OF ENERGY

$$P \sim \frac{V^2}{R}$$

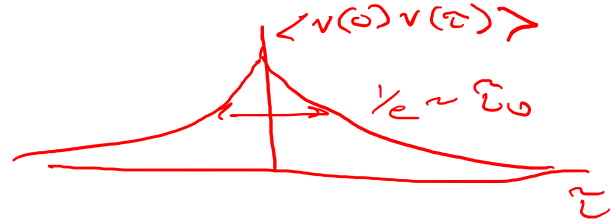
# POWER SPECTRAL DENSITY (PSD)

$$S_v(f) = \lim_{T\to\infty} \left\{ \frac{1}{T} \left| \tilde{V}(f) \right|^2 \right\} \quad \} \; V^2 \cdot s$$

$\tilde{V} \to V \cdot s$ $\qquad |\tilde{V}|^2 \quad V^2 \cdot s^{\cancel{2}} \cdot \frac{1}{\cancel{s}}$

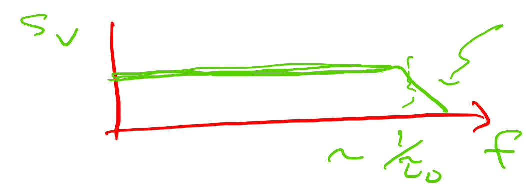
$\longrightarrow$ CAN ALSO DEFINE IN TERMS OF AUTOCORRELATION FUNCTION

$$S_v(f) = \int_{-\infty}^{\infty} \underline{\langle v(\overset{\xi}{t})\, v(t-\tau) \rangle}\, e^{-2\pi j f \tau}\, d\tau$$




$\Longrightarrow$ UNITS: $\underline{V^2/Hz}$

$$\longrightarrow \langle v^2(t) \rangle = \int_{-\infty}^{\infty} S_v(f)\, df$$



Proof:

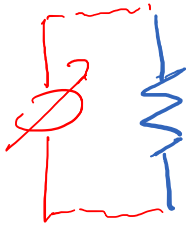
$$\langle v^2 \rangle = \lim_{T \to \infty} \left\{ \frac{1}{T} \int_{-T/2}^{T/2} v^2(t)\, dt \right\}$$

$$= \int_{-\infty}^{\infty} \lim_{T \to \infty} \left\{ \frac{1}{T} v^2(t) \right\} dt$$

$$= \int_{-\infty}^{\infty} \lim_{T \to \infty} \left\{ \frac{1}{T} \left| \tilde{V}(f) \right|^2 \right\} df$$

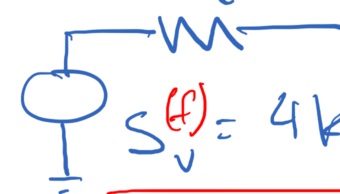
$$= \int_{-\infty}^{\infty} S_v(f)\, df$$

# Johnson / Nyquist Noise




R @ T

$\longrightarrow$




$V_T$

$S_V(f) = 4k_BTR$

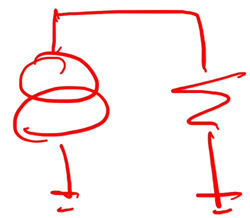
"WHITE" NOISE

TH ⟷ NORTON

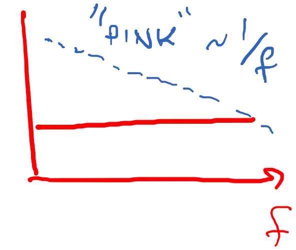
$I_N = \frac{V_T}{R}$

$\frac{A^2}{Hz}$

$S_I = \frac{4k_BT}{R}$

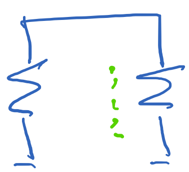


R



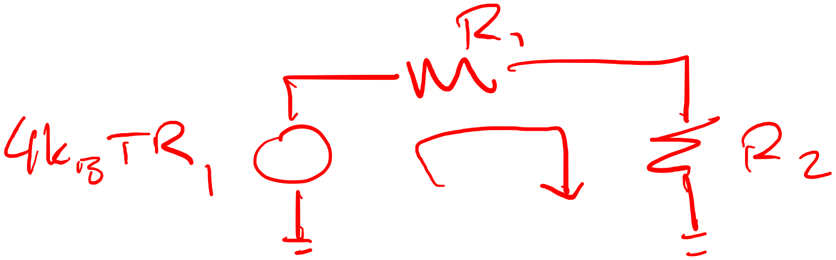

e.g.

$\rightarrow$ $R_1$ @ T



$R_2$ @ T = 0 ?

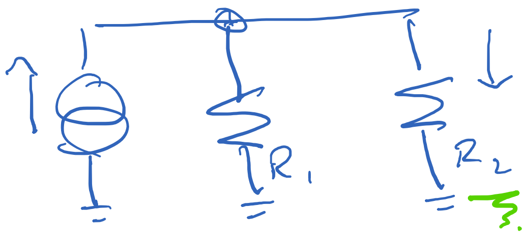
Noise current in $R_2$?

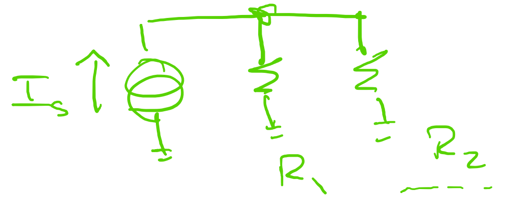



$$I_{loop} = \frac{v_s}{R_1 + R_2}$$

$$\boxed{S_I = \frac{4k_BTR_1}{(R_1+R_2)^2}}$$

$$S_I = \frac{4kT}{R_1}$$



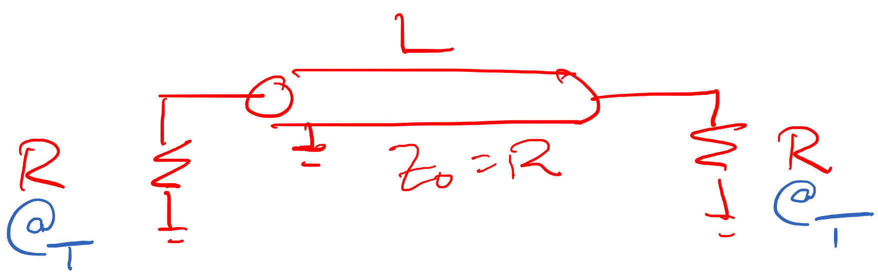

$$I(R_2) = I_s \cdot \frac{1/R_2}{1/R_1 + 1/R_2}$$

$$= I_s \frac{R_1}{R_1 + R_2}$$

$$S_I\Big|_{R_2} = \frac{4k_BT}{R_1} \cdot \frac{R_1^2}{(R_1+R_2)^2}$$

$$\boxed{S_I = \frac{4k_BTR_1}{(R_1+R_2)^2}}$$

# Johnson Noise as 1-D Realization of Blackbody Radiation




Line supports standing waves w/ $\lambda = \frac{2L}{n}$ ;

$$f_n = \frac{cn}{2L}$$

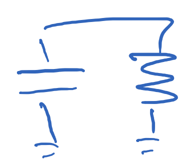
$c \rightarrow$ prop. speed

∴ Density of modes

$$\sigma(f) = 2L/c$$

Planck Distribution

$$\langle E(f) \rangle = \frac{hf}{e^{hf/k_B T} - 1} \approx k_B T$$

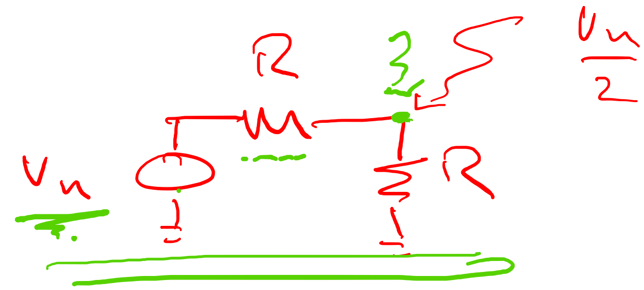


EACH RESISTOR CONTRIBUTES 1/2 POWER

NOISE POWER FROM EACH RESISTOR:

$\longrightarrow \frac{\text{ENERGY}}{\text{MODE}} \times$ # OF MODES $\div$ PROPAGATION TIME

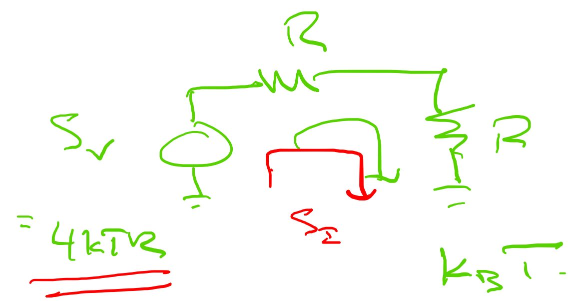
$$\frac{1}{2}k_BT \times \frac{2L}{c}\Delta f \div \frac{L}{c}$$




$$\Longrightarrow P_N = k_BT\Delta f$$

$$k_BT\Delta f = \frac{\langle v_n^2\rangle}{4R}$$

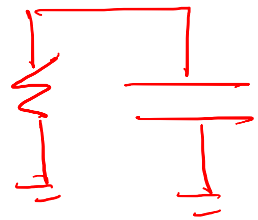
$$\langle v_n^2\rangle = 4k_BTR\Delta f$$




$$\longrightarrow S_V(f) = 4k_BTR$$

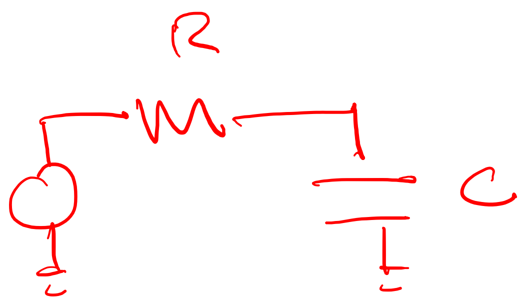
$S_I = \frac{4kTR}{4R^2} = \frac{kT}{R}$ ; $\frac{\text{POWER/BW}}{k_BT}$

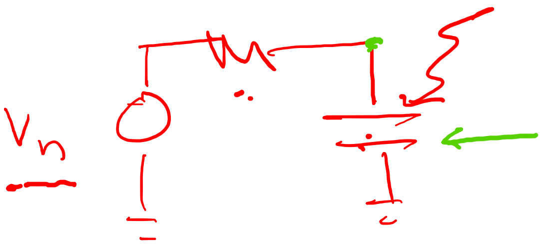
R@T



WHAT IS VOLTAGE NOISE ON CAP?

4kTR






$$A(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1+j\omega\tau}$$

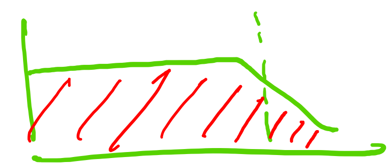
$\tau \equiv RC$

$$\rightarrow |A|^2 = \frac{1}{1+\omega^2\tau^2}.$$

$$S_v\Big|_{CAP} = 4kTR\cdot|A|^2 = \frac{4kTR}{1+\omega^2\tau^2}$$

WHAT IS $\langle v^2\rangle$ ON CAP. ?? $\rightarrow \int S_v\,df$.

NOISE BANDWIDTH $\leftrightarrow \int |A(f)|^2\,df$.



$1/\tau = 1/RC$

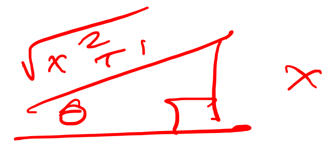
$$\rightarrow \int_0^\infty \frac{1}{1+(2\pi f\tau)^2}\,df$$

$x = 2\pi\tau f$

$dx = 2\pi\tau\,df$

$$= \frac{1}{2\pi\tau}\int_0^\infty \frac{dx}{1+x^2}$$

$$\int |A(f)|^2\, df = \frac{1}{2\pi\tau} \int_0^\infty \frac{dx}{1+x^2}$$

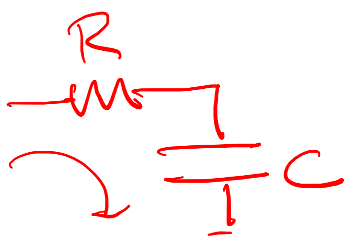



$$x = \tan\theta$$

$$dx = \frac{1}{\cos^2\theta}\, d\theta$$

$$= \frac{1}{2\pi\tau} \int_0^{\pi/2} \cos^2\theta \, \frac{1}{\cos^2\theta}\, d\theta$$

$$= \frac{1}{4\tau} = \frac{1}{4RC}$$




NOISE BANDWIDTH $\frac{1}{4RC}$