SAMPLING THEORY

ANALOG VOLTAGE $v_a(t)$

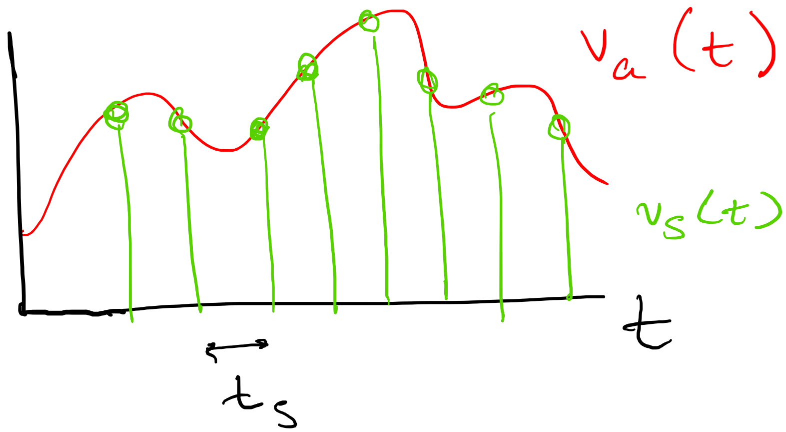
FOURIER TRANSFORM $g_a(\omega)$

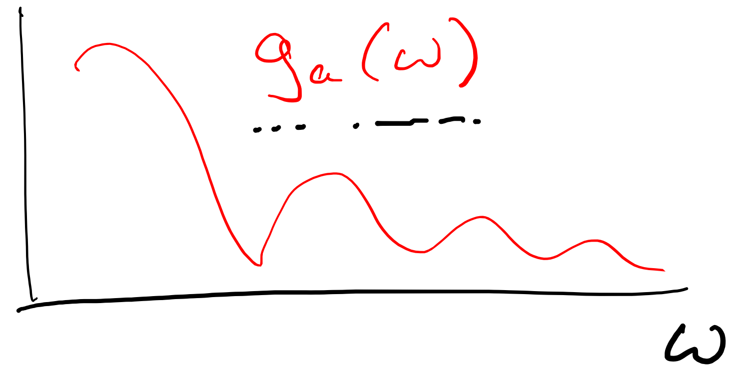
$$v_a(t) = \frac{1}{\sqrt{2\pi}} \int g_a(\omega) e^{j\omega t} d\omega$$

$$g_a(\omega) = \frac{1}{\sqrt{2\pi}} \int v_a(t) e^{-j\omega t} dt$$

SAMPLE v_a @ INTERVALS t_s

WHAT IS SPECTRAL CONTENT OF SAMPLED WAVEFORM?





$$v_s(t) = \frac{1}{\sqrt{2\pi}} \int g_s(\omega)\, e^{j\omega t}\, d\omega$$

$$g_s(\omega) = \frac{1}{\sqrt{2\pi}} \int v_s(t)\, e^{-j\omega t}\, dt$$

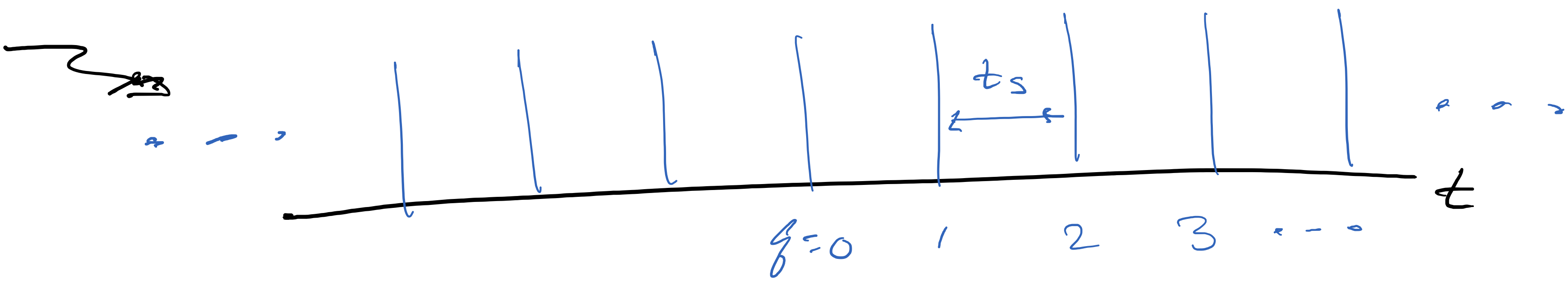
SAMPLED WAVEFORM

??

INTRODUCE

$$\Delta(t\,;\,t_s) = \sum_{\xi=-\infty}^{\infty} \delta(t - \xi t_s)\, t_s$$

ξ INTEGER



THEN

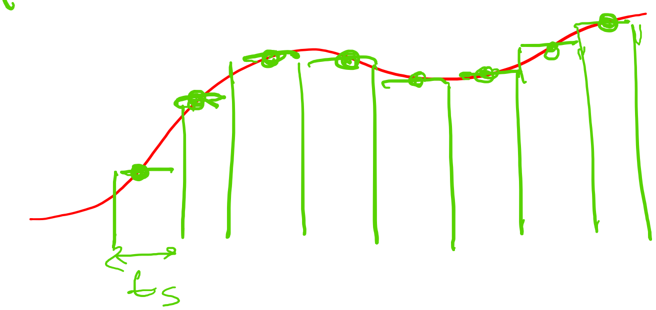
$$v_s(t) = v_a(t)\,\Delta(t\,;\,t_s)$$

$$= \sum_{\xi=-\infty}^{+\infty} v_a(t)\,\delta(t - t_s)\, t_s$$

e.g. $\int v_s(t)\,dt = \int \underbrace{\sum v_a(t)\,\delta(t-q t_s)\,t_s}\,dt$

$\int f(x)\,\delta(x-x_0)\,dx = f(x_0)$

$= \sum_q v_a(q t_s)\,t_s$



PIECEWISE INTEGRATION

NOW, $\Delta(t;t_s)$ IS A PERIODIC FUNCTION

→ CAN EXPRESS AS A DISCRETE FOURIER SERIES

RECALL

PERIODIC

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

FUNDAMENTAL

$\omega_0 = \frac{2\pi}{T}$

PERIOD

WHERE

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,e^{-jn\omega_0 t}\,dt$$

PERIOD

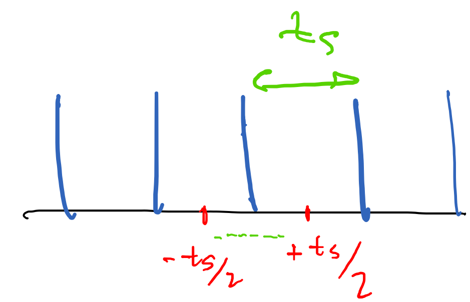
[ORTHOGONALITY OF $e^{jn\omega_0 t}$]

$c_n^* = c_{-n}$

$\therefore$ WE CAN WRITE

$$\Delta(t;\, t_s) = \sum_{-\infty}^{\infty} a_n\, e^{j n \omega_s t} \;; \quad \omega_s = \frac{2\pi}{t_s}$$

[SAMPLING FREQ.]

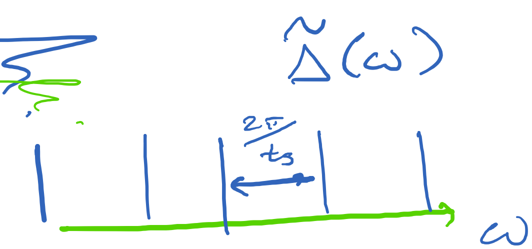


$$\rightarrow \quad a_n = \frac{1}{t_s} \int_{-t_s/2}^{t_s/2} \Delta(t;\, t_s)\, e^{-j n \omega_s t}\, dt$$

$$= \frac{1}{t_s} \int_{-t_s/2}^{t_s/2} \left[\sum \delta(t - q t_s)\, t_s \right] e^{-j n \omega_s t}\, dt$$

$$= 1/t_s \times t_s \int_{-t_s/2}^{+t_s/2} \delta(t)\, e^{-j n \omega_s t}\, dt = 1$$

$$\boxed{\Delta(t;\, t_s) = \sum_{-\infty}^{\infty} e^{j n \omega_s t}}$$



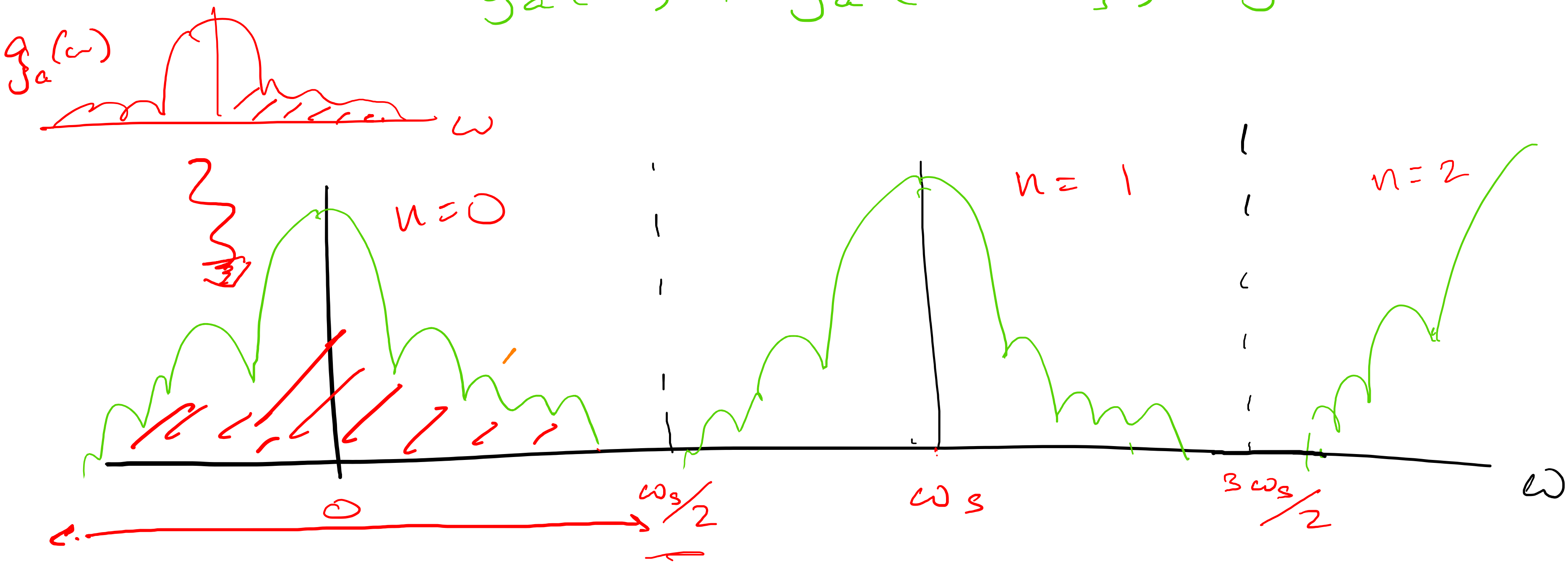
$$\therefore \quad g_s(\omega) = \frac{1}{\sqrt{2\pi}} \int \underbrace{v_a(t)\left[\sum e^{jn\omega_s t}\right]}_{v_a(t)\,\Delta(t;t_s) \,=\, v_s(t)} e^{-j\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \sum_n \int v_a(t)\, e^{-j(\omega - n\omega_s)t}\,dt$$

$$\rightarrow \quad g_s(\omega) = \underbrace{\sum_{n=-\infty}^{\infty} g_a(\omega - n\omega_s)}$$

SUM OF COPIES OF ORIGINAL FOURIER TRANSFORM $g_a(\omega)$ DISPLACED IN FREQUENCY BY INTEGER MULTIPLES OF SAMPLING FREQUENCY ω_s.

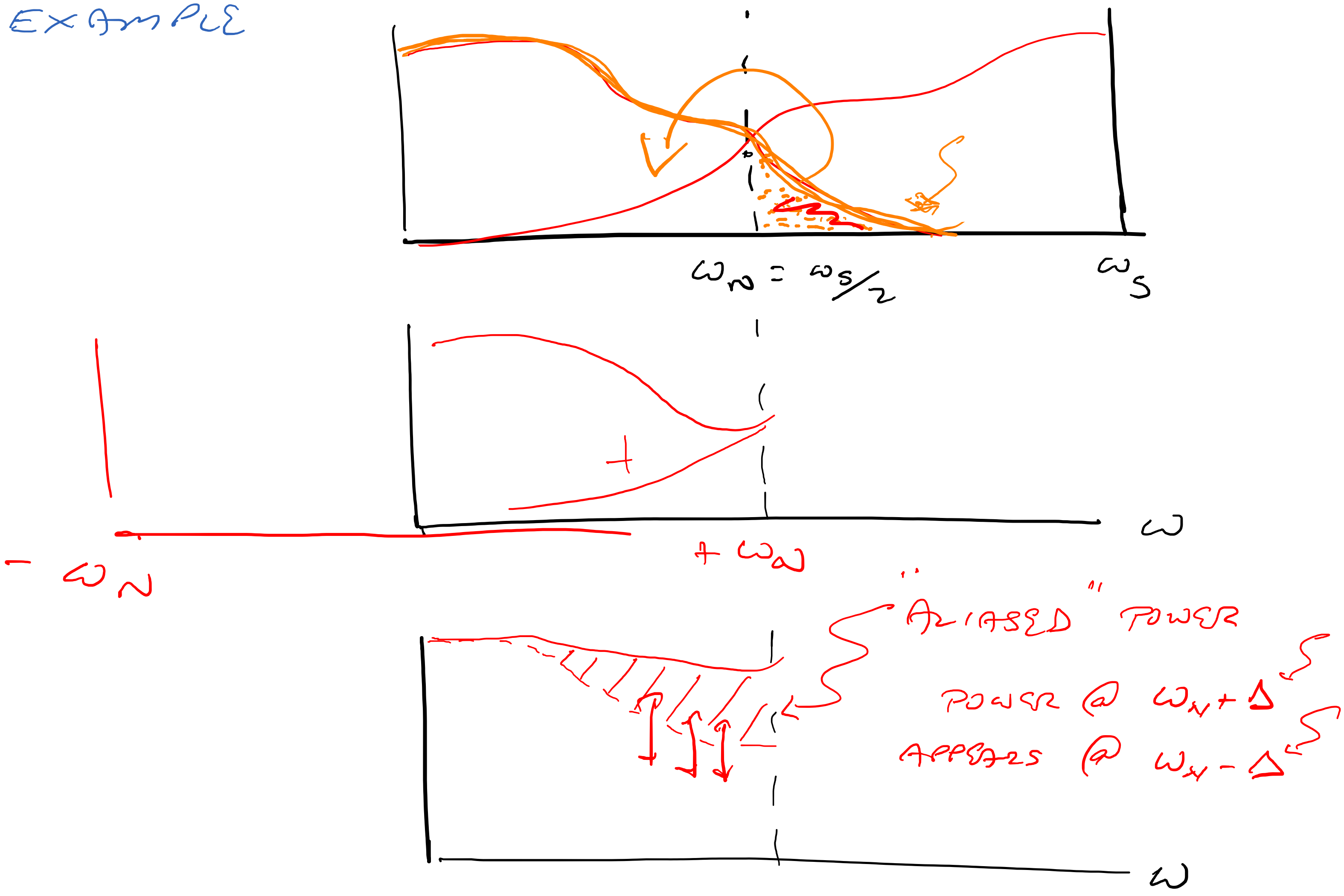
i.e. $g_s(\omega) = \cdots g_a(\omega - 2\omega_s) + g_a(\omega - \omega_s) +$
$g_a(\omega) + g_a(\omega + \omega_s) + g_a(\omega + 2\omega_s) + \cdots$

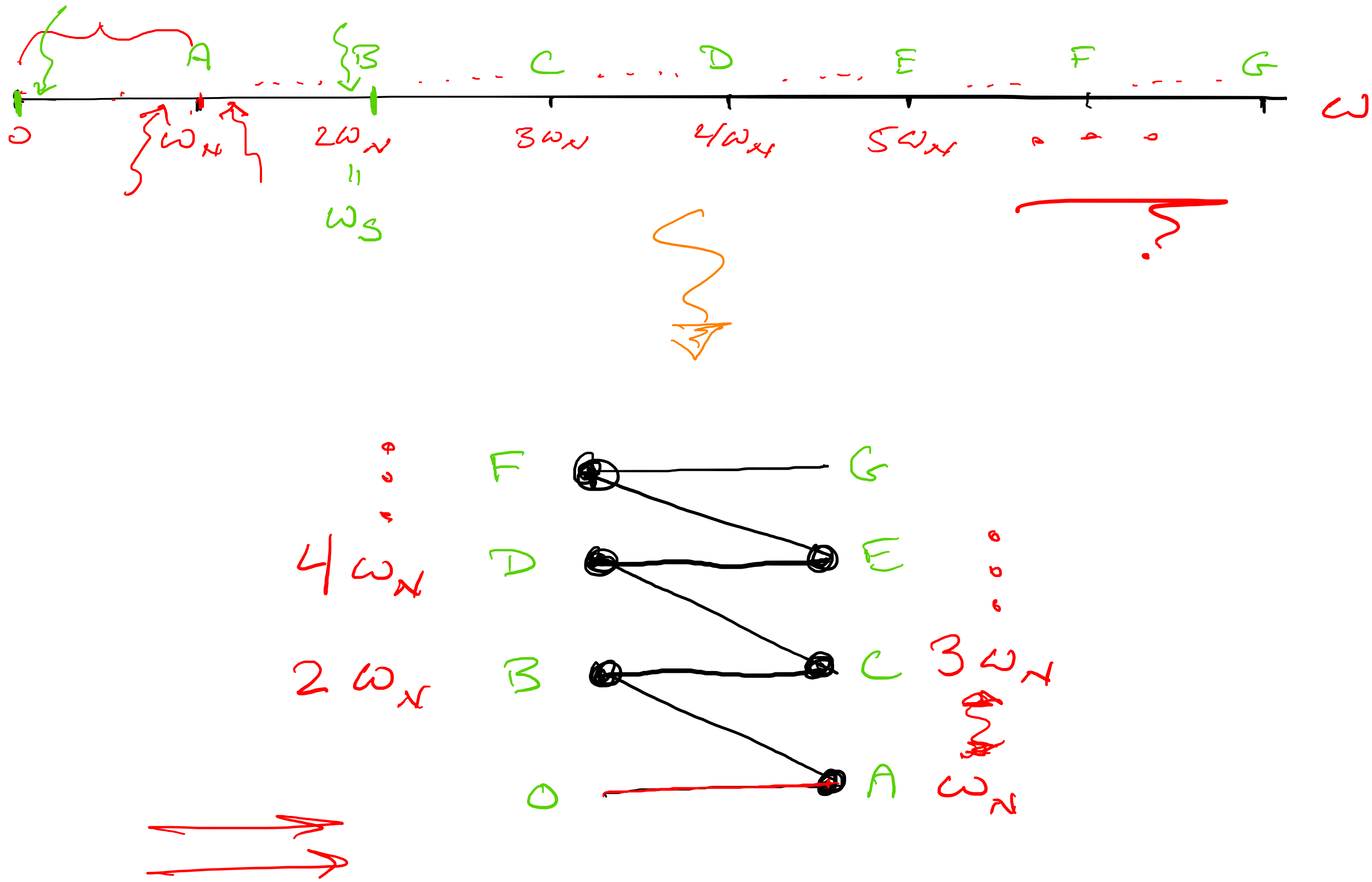


$\omega_s/2 \longrightarrow$ NYQUIST FREQUENCY

IF NO SPECTRAL CONTENT @ $\omega > \omega_s/2$, SAMPLING FAITHFULLY REPRODUCES "TRUE" SPECTRUM.

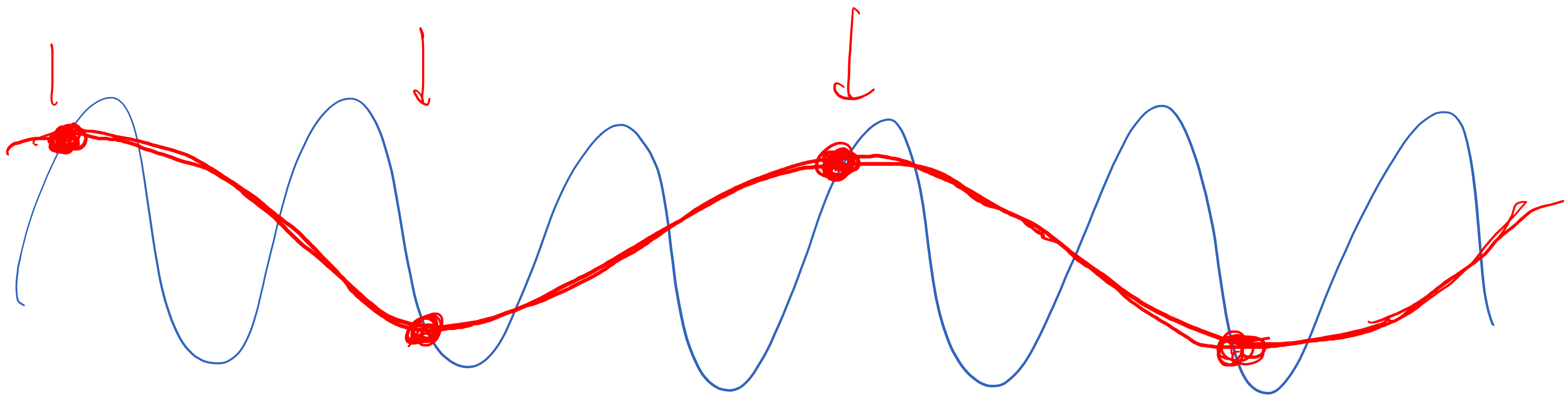
ANOTHER EXAMPLE





A
B
C
D
E
F
G
0
ω_N
$2\omega_N$
ω_s
$3\omega_N$
$4\omega_N$
$5\omega_N$
ω
F
D
B
0
G
E
C
A
$4\omega_N$
$2\omega_N$
$3\omega_N$
ω_N

EXAMPLE



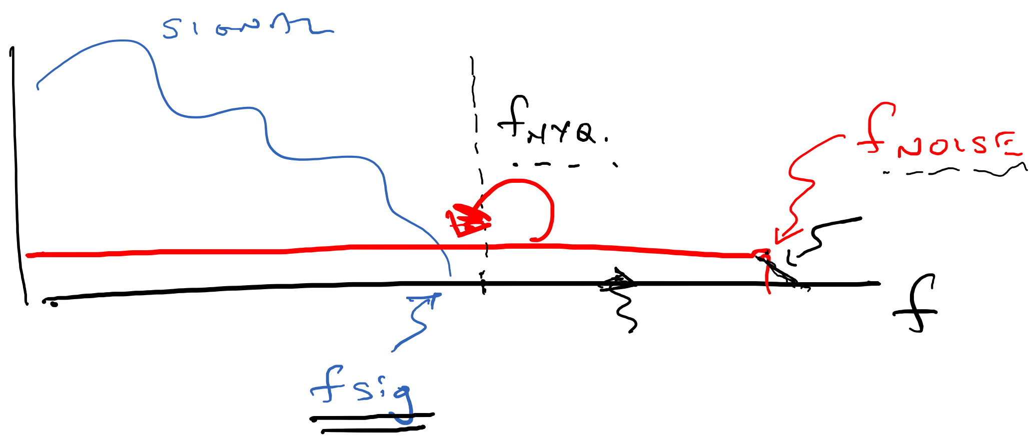
SIGNAL @ $\frac{3}{2} f_{SAMPLE} = 3 f_{NYQUIST}$ IS ALIASED

TO $f_{NYQUIST} = \frac{1}{2} f_{SAMPLE}$

EXAMPLE:

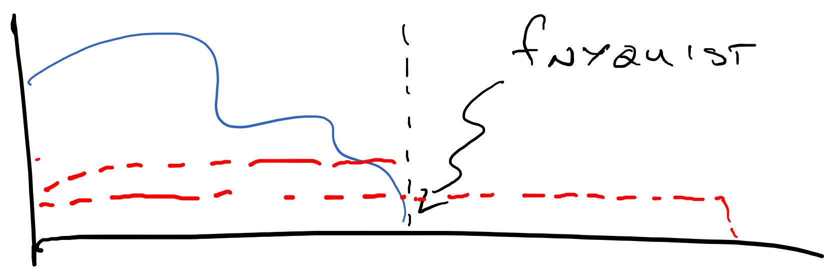
SIGNAL IN BW $0 \rightarrow f_{sig}$.

AMPLIFIER w/ [WHITE] NOISE
IN BW $0 \rightarrow f_{Noise} > f_{sig}$

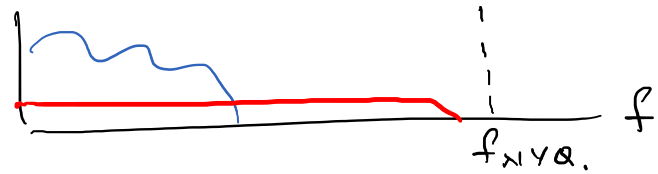


SAMPLE @ $2f_{sig}$?

SNR DEGRADATION
BY FACTOR 2



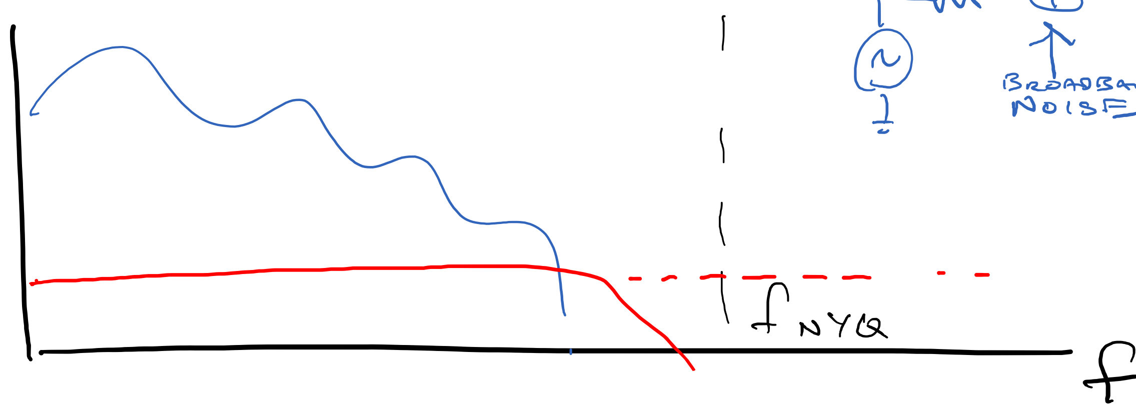
SOLUTIONS:

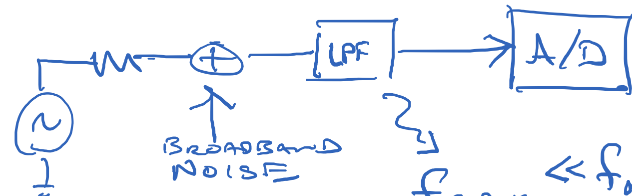


① SAMPLE @ FREQUENCY S.T. $f_{NYQUIST} > f_{NOISE}$

[NO NOISE ABOVE $f_{NYQ.}$ TO GET ALIASED DOWN TO LOW FREQUENCY]

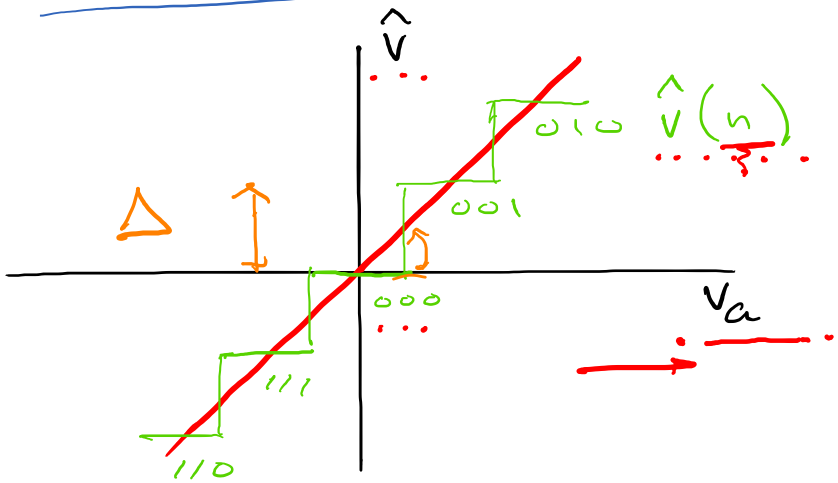
② ANTI-ALIASING FILTER.





$f_{CORNER} \ll f_{NYQ}$

FINITE SAMPLING RESOLUTION : QUANTIZATION ERROR



$$v_a(t) \longrightarrow \boxed{\text{SAMPLE}} \xrightarrow{v_s(n)} \boxed{\text{QUANTIZE}} \longrightarrow \hat{v}(n)$$

$$\hat{v}(n) = Q[v_s(n)]$$

NONLINEAR TRANSFORMATION

SIMPLIFIED STATISTICAL / NOISE MODEL

$$v_a(t) \longrightarrow \boxed{\text{SAMPLE}} \xrightarrow{v_s(n)} \bigoplus \longrightarrow \hat{v} = v_a + v_e$$

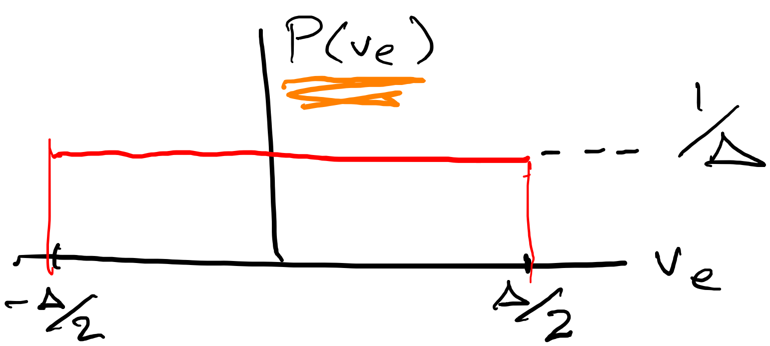
$$\uparrow$$

$$v_e(n)$$

$$-\frac{\Delta}{2} \le v_e \le \frac{\Delta}{2}$$

FOR STATISTICAL ANALYSIS OF SAMPLED SIGNALS, WE'LL ASSUME:

1. V_e IS A SAMPLE SEQUENCE OF A STATIONARY RANDOM PROCESS
2. ERROR SEQUENCE V_e IS UNCORRELATED W/ EXACT SEQUENCE OF SIGNAL SAMPLES V_S
3. RANDOM SAMPLES OF THE ERROR SEQUENCE ARE UNCORRELATED (WHITE NOISE)
4. PROBABILITY DISTRIBUTION OF ERROR PROCESS IS UNIFORM OVER RANGE OF QUANTIZATION ERROR.



$\Rightarrow$ ASSUMPTIONS ARE BEST SATISFIED FOR MORE COMPLEX SIGNALS

[FOR WHICH CORRELATION BETWEEN V_a & V_e DECREASES]

SPECTRAL DENSITY OF QUANTIZATION NOISE??

RECALL $\displaystyle\int_{-\infty}^{\infty} S_x(f)\,df = \lim_{T\to\infty}\left\{\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt\right\}$

$$= \langle x^2 \rangle$$

HERE, WANT $S_e(f)$

CALCULATE $\displaystyle\langle v_e^2\rangle = \int_{-\Delta/2}^{\Delta/2} P(v_e)\,v_e^2\,dv_e = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} v_e^2\,dv_e$

$$\langle v_e^2\rangle = \frac{1}{3\Delta}\left[\frac{\Delta^3}{8} - \left(-\frac{\Delta^3}{8}\right)\right] = \frac{1}{12}\Delta^2$$

$$\therefore \int_{-f_{NYQ}}^{f_{NYQ}} S_e(f)\,df = \frac{1}{12}\Delta^2$$

$$\rightarrow S_e(f) \cdot 2f_{NYQ} = \frac{1}{12}\Delta^2$$

$$f_{NYQ} = \frac{1}{2t_s}$$

$$\rightarrow \boxed{\rightarrow S_e(f) = \frac{\Delta^2}{12} t_s}$$

→ DECREASE QUANTIZATION NOISE BY

EITHER DECREASING Δ

OR DECREASING t_s