SAMPLING THEORY

ANALOG VOLTAGE $V_a(t)$

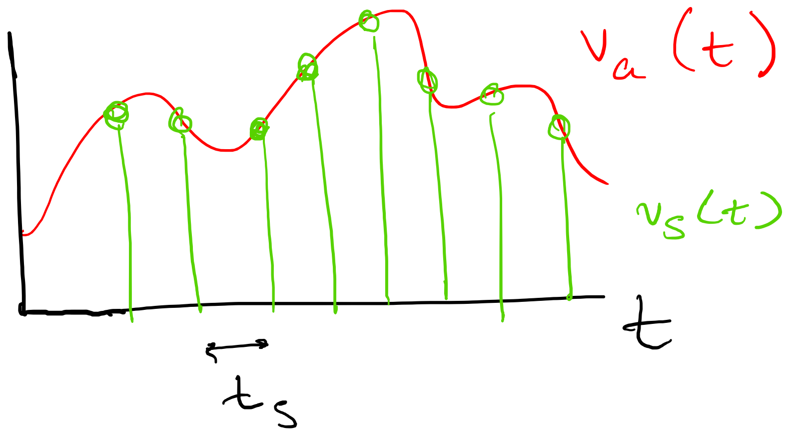
FOURIER TRANSFORM $g_a(\omega)$

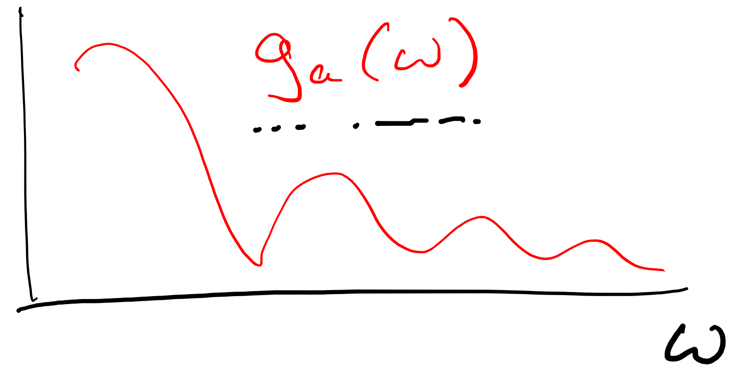
$$V_a(t) = \frac{1}{\sqrt{2\pi}} \int g_a(\omega) e^{j\omega t} \, d\omega$$

$$g_a(\omega) = \frac{1}{\sqrt{2\pi}} \int V_a(t) e^{-j\omega t} \, dt$$

SAMPLE V_a @ INTERVALS t_s

WHAT IS SPECTRAL CONTENT OF SAMPLED WAVEFORM?





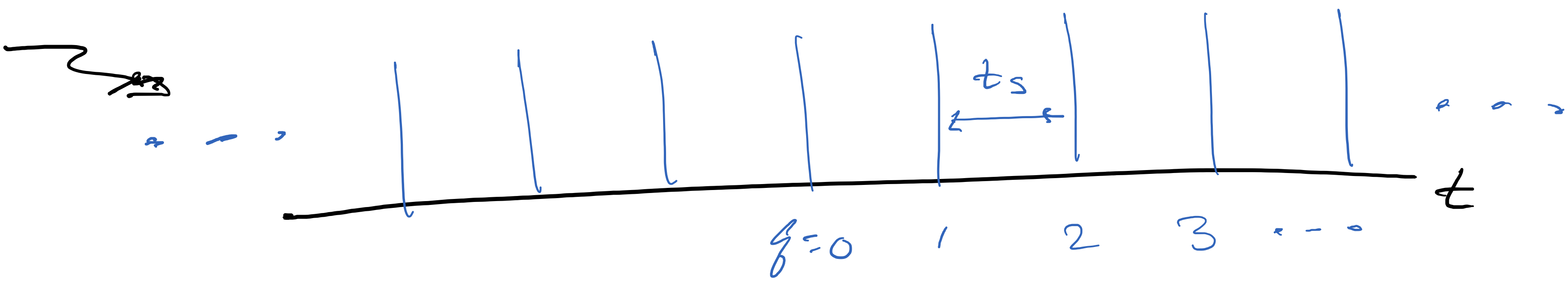
$$\left.\begin{aligned} v_s(t) &= \frac{1}{\sqrt{2\pi}} \int g_s(\omega)\, e^{j\omega t}\, d\omega \\ g_s(\omega) &= \frac{1}{\sqrt{2\pi}} \int v_s(t)\, e^{-j\omega t}\, dt \end{aligned}\right\}$$ SAMPLED WAVEFORM

??

INTRODUCE

$$\Delta(t\,;\,t_s) = \sum_{\xi=-\infty}^{\infty} \delta(t-\xi t_s)\, t_s$$

ξ INTEGER



THEN

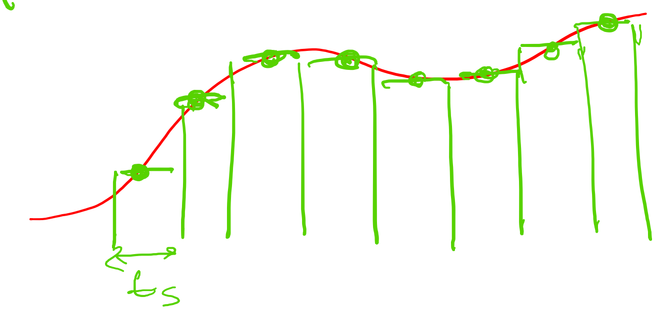
$$v_s(t) = v_a(t)\,\Delta(t\,;\,t_s)$$

$$= \sum_{\xi=-\infty}^{+\infty} v_a(t)\,\delta(t-t_s)\,t_s$$

e.g. $\int v_s(t)\,dt = \int \sum v_a(t)\,\delta(t - q t_s)\, t_s \, dt$

$\int f(x)\,\delta(x-x_0)\,dx = f(x_0)$

$$= \sum_q v_a(q t_s)\, t_s$$



PIECEWISE INTEGRATION

~~thus~~

Now, $\Delta(t; t_s)$ is a PERIODIC FUNCTION

→ CAN EXPRESS AS A DISCRETE FOURIER SERIES

RECALL

PERIODIC

$$\rightarrow f(t) = \sum_{-\infty}^{\infty} c_n e^{j n \omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

FUNDAMENTAL

PERIOD

WHERE

$$\rightarrow c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, e^{-j n \omega_0 t}\, dt \qquad \left[\text{ORTHOGONALITY OF } e^{j n \omega_0 t}\right]$$

PERIOD

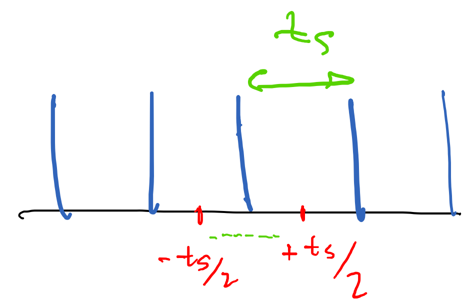
$$c_n^* = c_{-n}$$

∴ WE CAN WRITE

$$\Delta(t;t_s) = \sum_{-\infty}^{\infty} a_n e^{jn\omega_s t} \quad ; \quad \omega_s = \frac{2\pi}{t_s}$$

[SAMPLING FREQ.]

$$\rightarrow a_n = \frac{1}{t_s}\int_{-t_s/2}^{t_s/2} \Delta(t;t_s)\, e^{-jn\omega_s t}\, dt$$

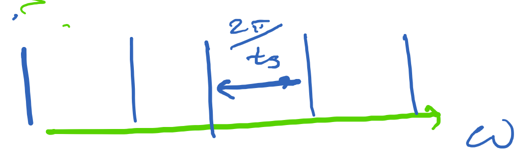


$$= \frac{1}{t_s}\int_{-t_s/2}^{t_s/2} \left[\sum \delta(t - q t_s)\, t_s\right] e^{-jn\omega_s t}\, dt$$

$$= 1/t_s \times t_s \int_{-t_s/2}^{+t_s/2} \delta(t)\, e^{-jn\omega_s t}\, dt = 1$$

$$\boxed{\Delta(t;t_s) = \sum_{-\infty}^{\infty} e^{jn\omega_s t}}$$

$\tilde{\Delta}(\omega)$



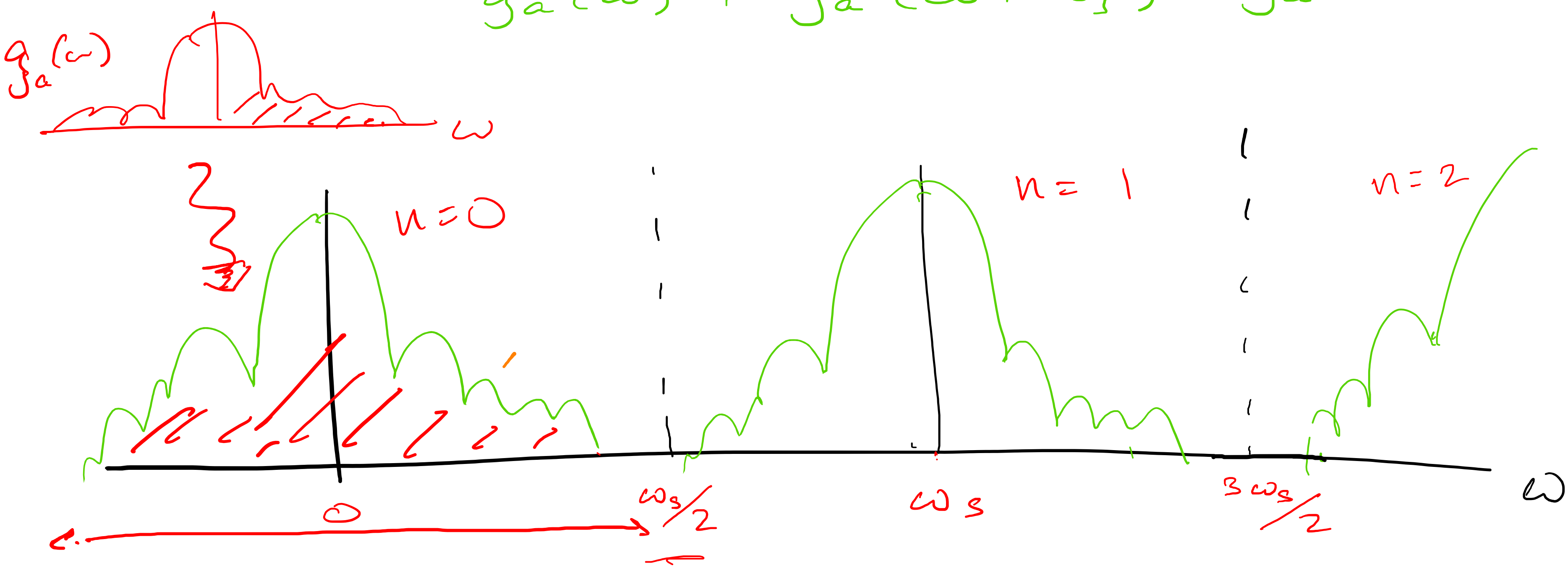
$$\therefore \quad g_s(\omega) = \frac{1}{\sqrt{2\pi}} \int \underbrace{v_a(t)\left[\sum e^{jn\omega_s t}\right]}_{\text{"}v_a(t)\,\Delta(t;t_s) = v_s(t)} e^{-j\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \sum_n \int v_a(t)\, e^{-j(\omega - n\omega_s)t}\,dt$$

$$\rightarrow \quad g_s(\omega) = \sum_{n=-\infty}^{\infty} g_a(\omega - n\omega_s)$$

SUM OF COPIES OF ORIGINAL FOURIER TRANSFORM $g_a(\omega)$ DISPLACED IN FREQUENCY BY INTEGER MULTIPLES OF SAMPLING FREQUENCY ω_s.

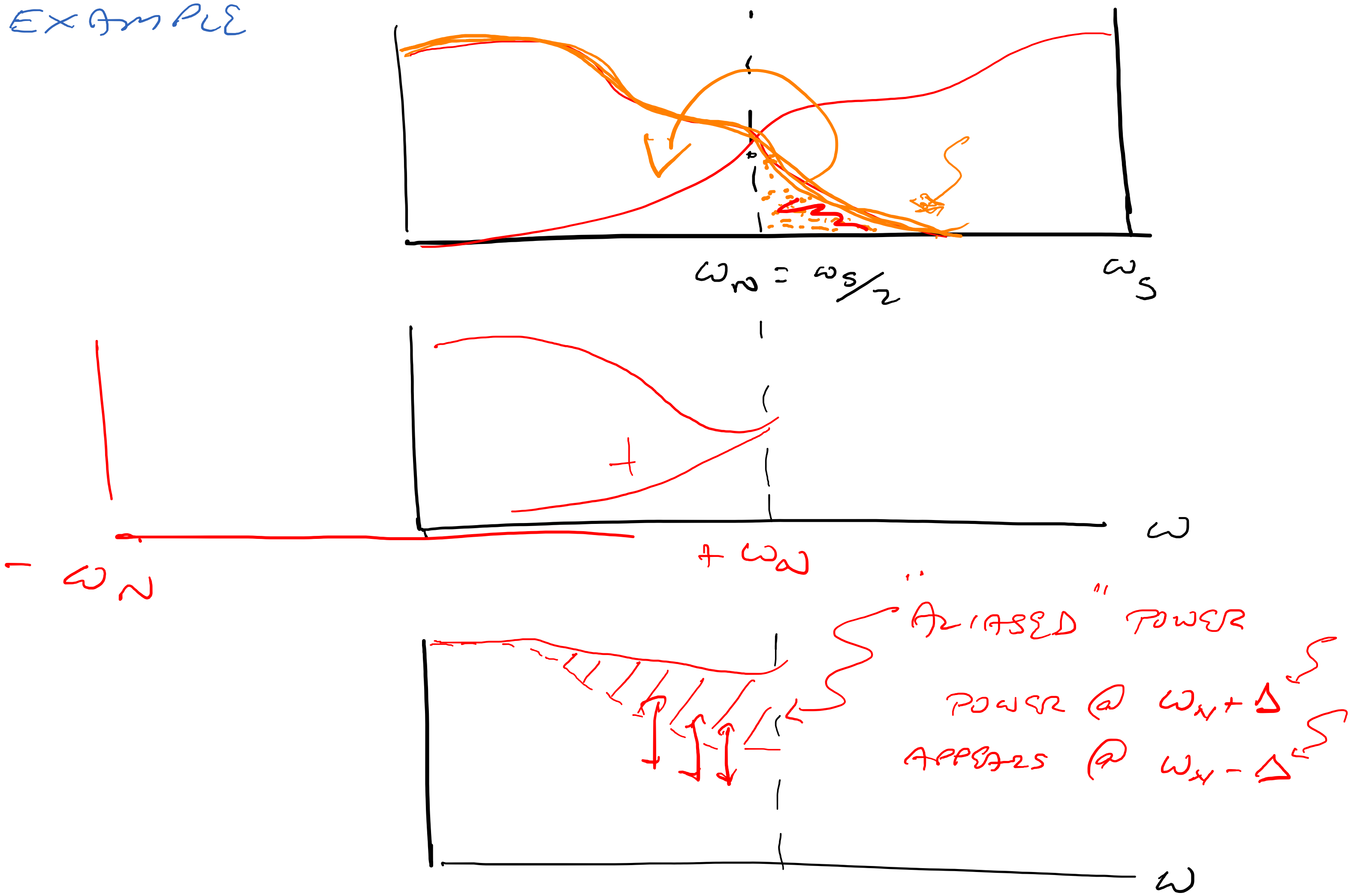
i.e. $g_s(\omega) = \cdots g_a(\omega - 2\omega_s) + g_a(\omega - \omega_s) +$
$g_a(\omega) + g_a(\omega + \omega_s) + g_a(\omega + 2\omega_s) + \cdots$

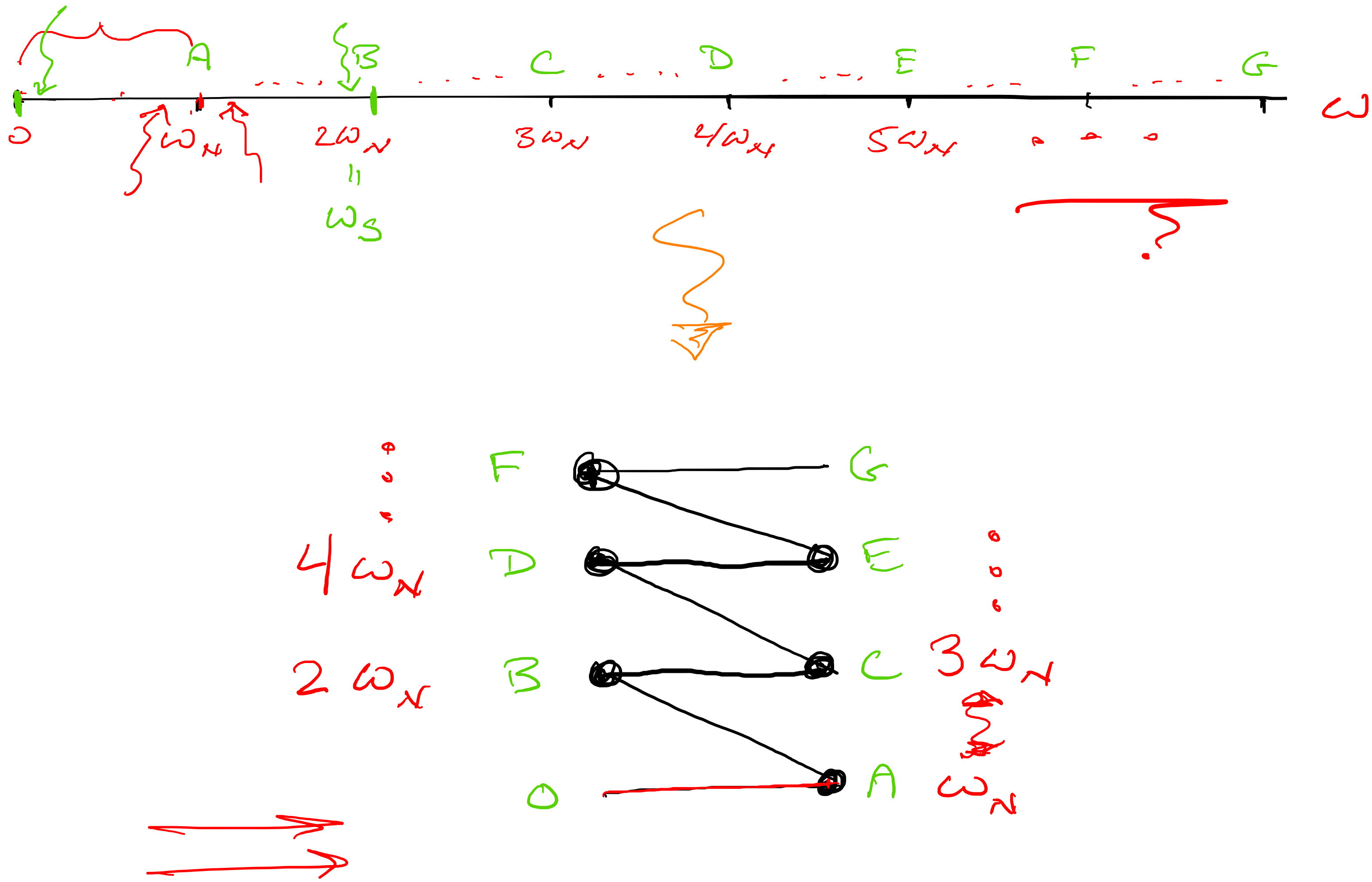


$\omega_s/2 \longrightarrow$ NYQUIST FREQUENCY

IF NO SPECTRAL CONTENT @ $\omega > \omega_s/2$, SAMPLING FAITHFULLY REPRODUCES "TRUE" SPECTRUM.

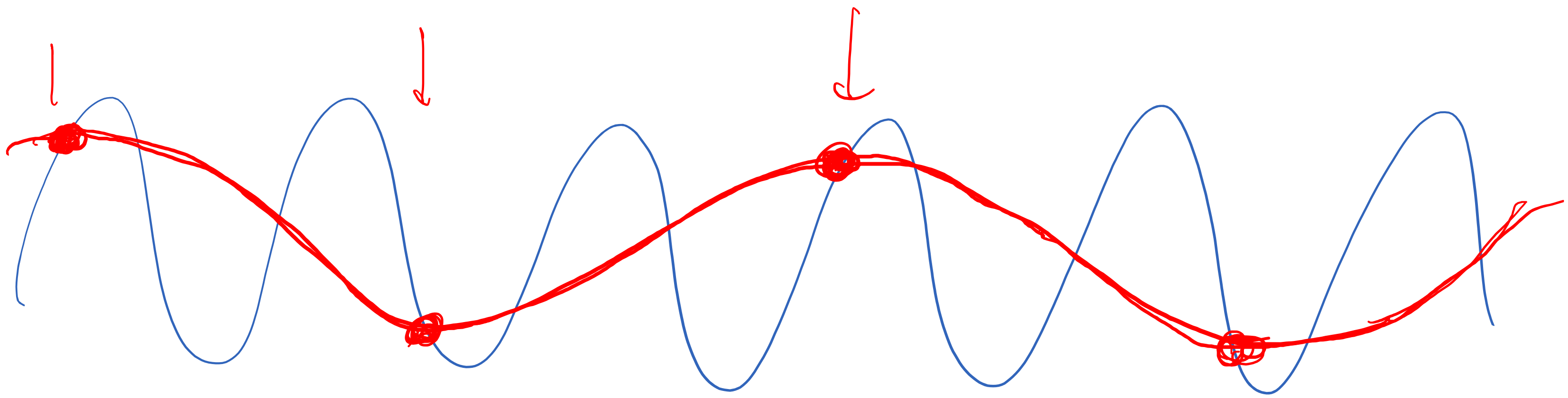
ANOTHER EXAMPLE





A
B
C
D
E
F
G
ω
0
ω_N
$2\omega_N$
ω_s
$3\omega_N$
$4\omega_N$
$5\omega_N$
F
G
$4\omega_N$
D
E
$2\omega_N$
B
C
$3\omega_N$
0
A
ω_N

EXAMPLE



SIGNAL @ $\frac{3}{2} f_{SAMPLE} = 3 f_{NYQUIST}$ IS ALIASED

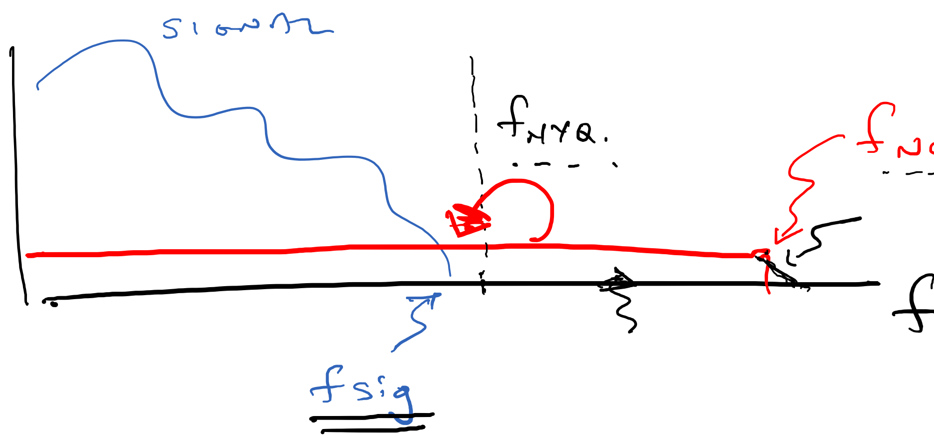
TO $f_{NYQUIST} = \frac{1}{2} f_{SAMPLE}$

EXAMPLE:

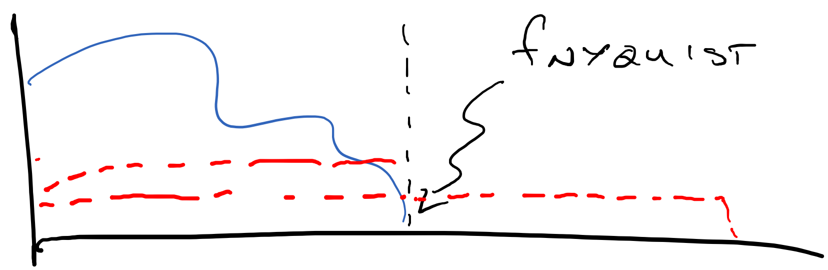
SIGNAL IN BW $0 \rightarrow f_{sig}$.

AMPLIFIER w/ [WHITE] NOISE

IN BW $0 \rightarrow f_{Noise} > f_{sig}$

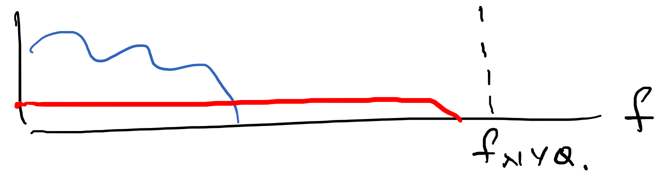


SAMPLE @ $2f_{sig}$?



SNR DEGRADATION BY FACTOR 2

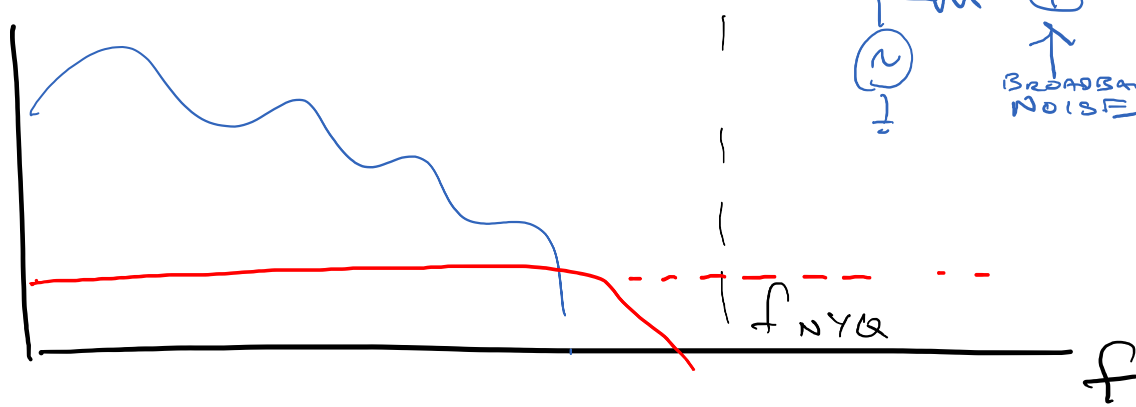
SOLUTIONS:

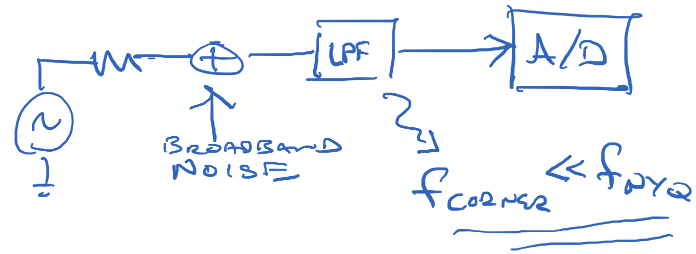


① SAMPLE @ FREQUENCY S.T. $f_{NYQUIST} > f_{NOISE}$

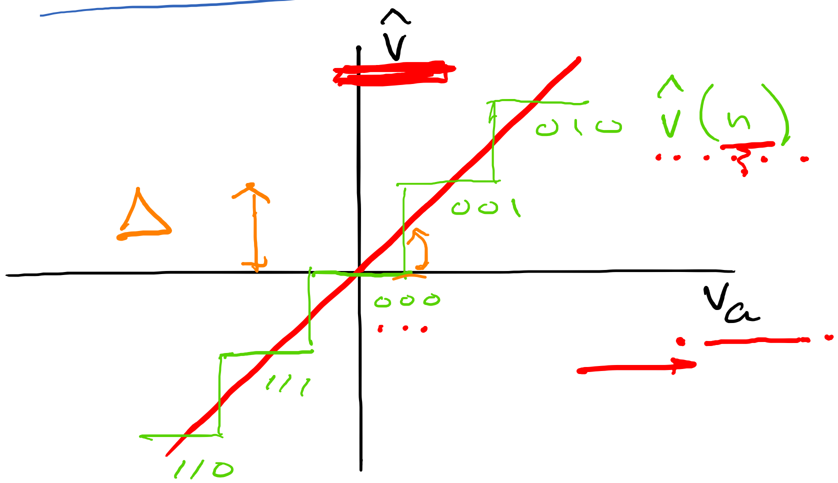
[NO NOISE ABOVE $f_{NYQ.}$ TO GET ALIASED DOWN TO LOW FREQUENCY]

② ANTI-ALIASING FILTER.





Finite Sampling Resolution : Quantization Error



$$v_a(t) \longrightarrow \boxed{\text{SAMPLE}} \xrightarrow{v_s(n)} \boxed{\text{QUANTIZE}} \longrightarrow \hat{v}(n)$$

$$\hat{v}(n) = Q\left[v_s(n)\right]$$

NONLINEAR TRANSFORMATION

SIMPLIFIED STATISTICAL / NOISE MODEL

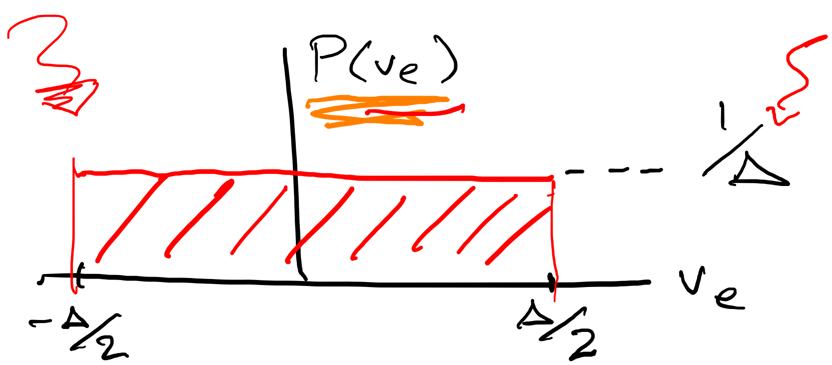
$$v_a(t) \longrightarrow \boxed{\text{SAMPLE}} \xrightarrow{v_s(n)} \oplus \longrightarrow \hat{v} = v_a + v_e$$

$$v_e(n) \uparrow$$

$$-\frac{\Delta}{2} \le v_e \le \frac{\Delta}{2}$$

FOR STATISTICAL ANALYSIS OF SAMPLED SIGNALS, WE'LL ASSUME:

1. V_e IS A SAMPLE SEQUENCE OF A STATIONARY RANDOM PROCESS
2. ERROR SEQUENCE V_e IS UNCORRELATED W/ EXACT SEQUENCE OF SIGNAL SAMPLES V_S
3. RANDOM SAMPLES OF THE ERROR SEQUENCE ARE UNCORRELATED (WHITE NOISE)
4. PROBABILITY DISTRIBUTION OF ERROR PROCESS IS UNIFORM OVER RANGE OF QUANTIZATION ERROR.



$\Rightarrow$ ASSUMPTIONS ARE BEST SATISFIED FOR MORE COMPLEX SIGNALS

[FOR WHICH CORRELATION BETWEEN V_a & V_e DECREASES]

SPECTRAL DENSITY OF QUANTIZATION NOISE ??

RECALL $$\int_{-\infty}^{\infty} S_x(f)\,df = \lim_{T\to\infty}\left\{\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt\right\}$$

$$= \langle x^2 \rangle$$

HERE, WANT $S_e(f)$

CALCULATE $$\langle v_e^2 \rangle = \int_{-\Delta/2}^{\Delta/2} P(v_e)\, v_e^2\, dv_e = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} v_e^2\, dv_e$$

$$\langle v_e^2 \rangle = \frac{1}{3\Delta}\left[\frac{\Delta^3}{8} - \left(-\frac{\Delta^3}{8}\right)\right] = \frac{1}{12}\Delta^2$$

LSB

$$\therefore \int_{-f_{NYQ}}^{f_{NYQ}} S_e(f)\, df = \frac{1}{12}\Delta^2$$

$$\rightarrow S_e(f) \cdot 2 f_{NYQ} = \frac{1}{12}\Delta^2$$

$$f_{NYQ} = \frac{1}{2t_s}$$

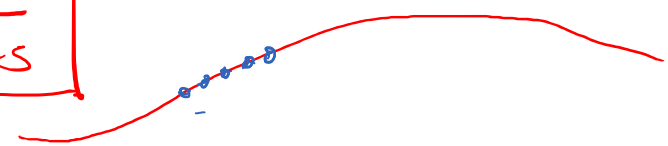
$$\rightarrow \boxed{\rightarrow S_e(f) = \frac{\Delta^2}{12} t_s}$$

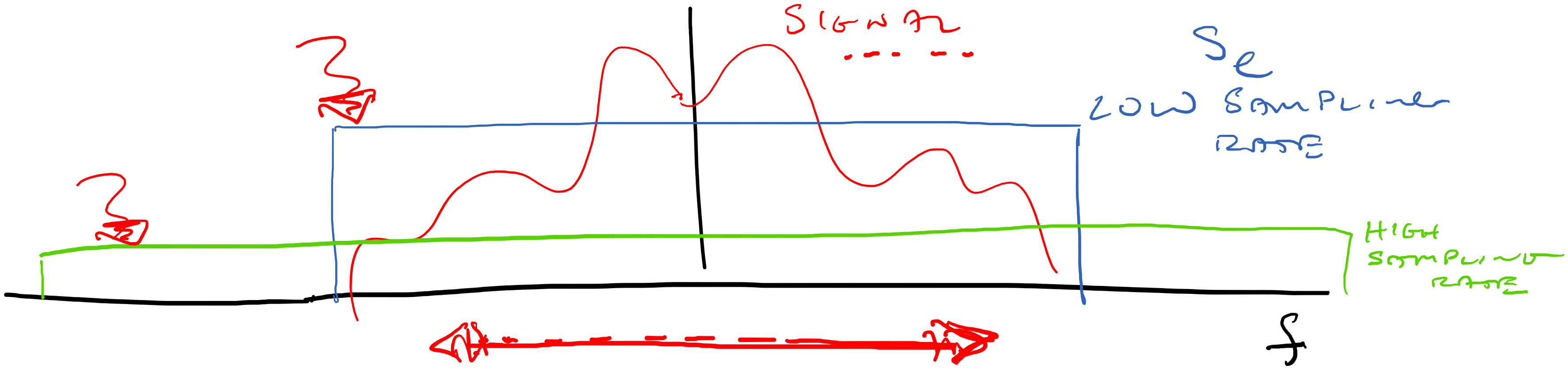
→ DECREASE QUANTIZATION NOISE BY

EITHER DECREASING Δ

OR DECREASING t_s

$$S_e(f) = \frac{\Delta^2}{12} t_s \qquad \boxed{\text{POWER SNR} \sim \frac{1}{t_s}}$$





→ "OVERSAMPLING"

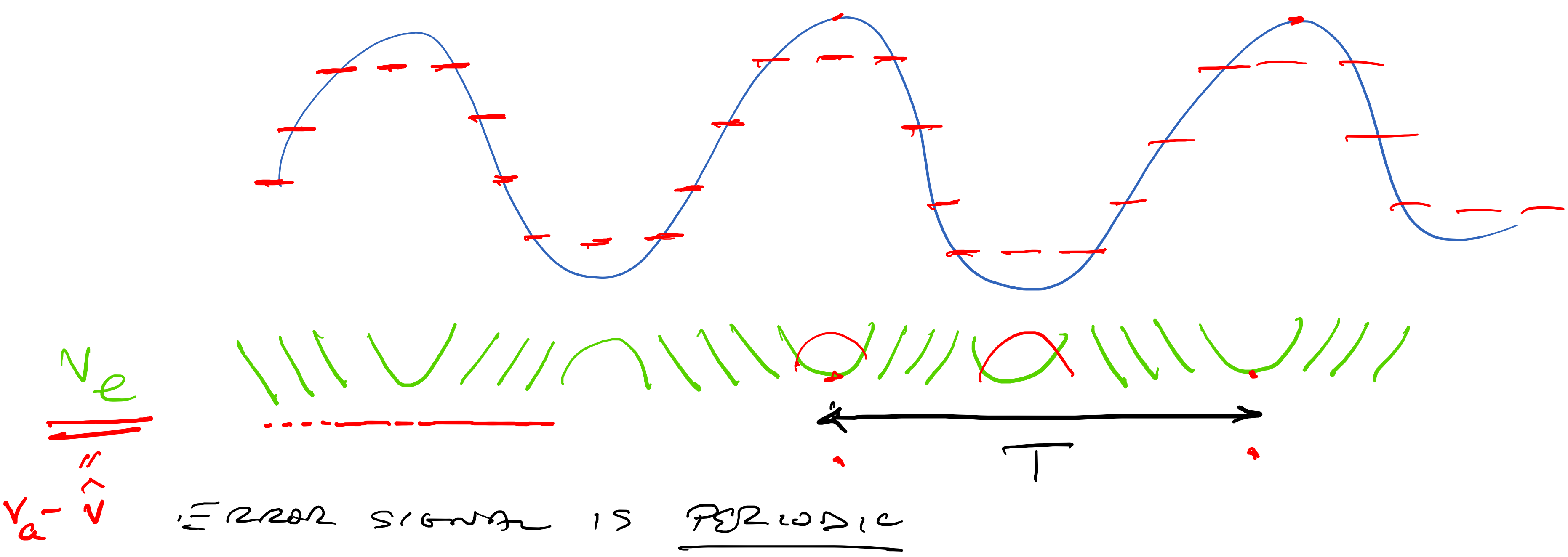
e.g. AD7760 Σ-Δ ADC

2.5 MSPS, 24 BIT

→ SAMPLE @ 40 MHz, THEN USE DIGITAL FILTER TO ELIMINATE OUT-OF-BAND QUANTIZATION NOISE

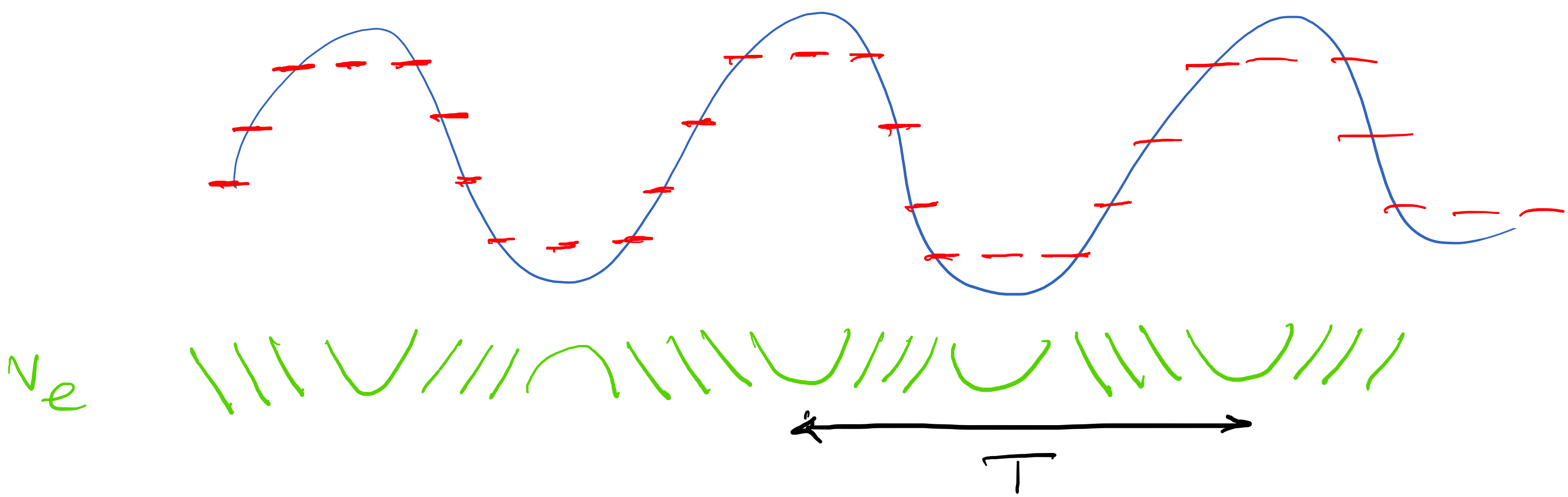
ANOTHER CONSEQUENCE OF FINITE A/D RESOLUTION:
HARMONIC DISTORTION

e.g., SINUSOIDAL INPUT

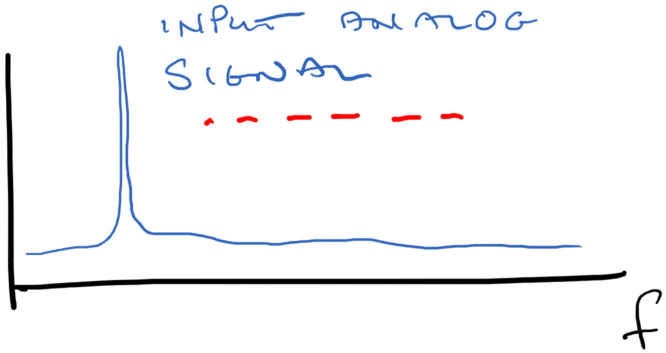


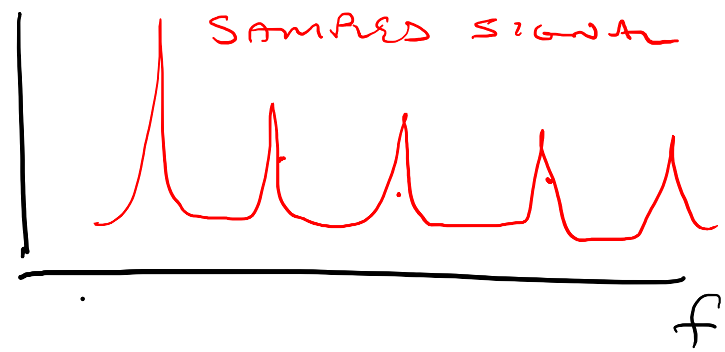
ERROR SIGNAL IS PERIODIC

→ SPURIOUS HARMONICS @ HIGH FREQUENCY [MULTIPLES OF INPUT FREQUENCY]



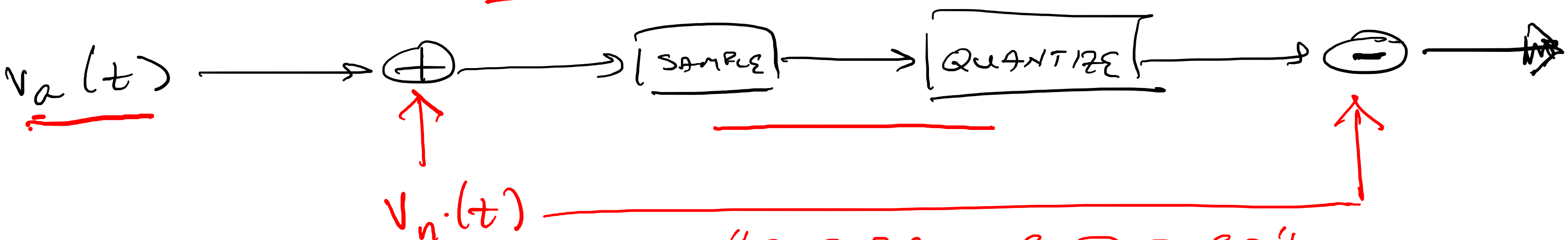
FOURIER DOMAIN:





SOLUTION ? ADD NOISE BEFORE QUANTIZATION

"DITHERING"

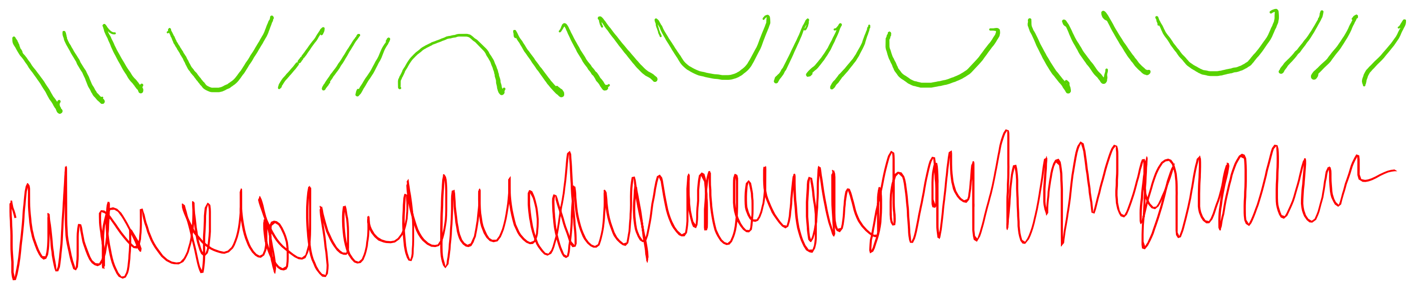


RMS AMPLITUDE

$\simeq$ 1 LSB

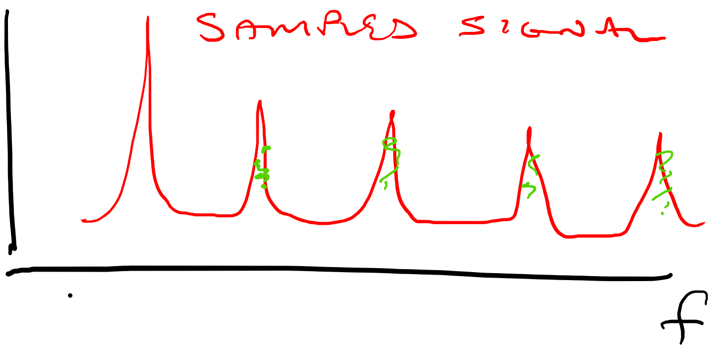
"SUBTRACTIVE DITHER"

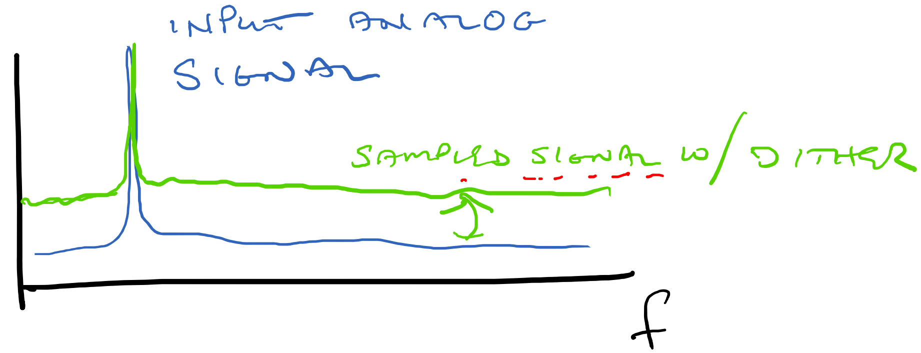
V_e



WITH DITHER

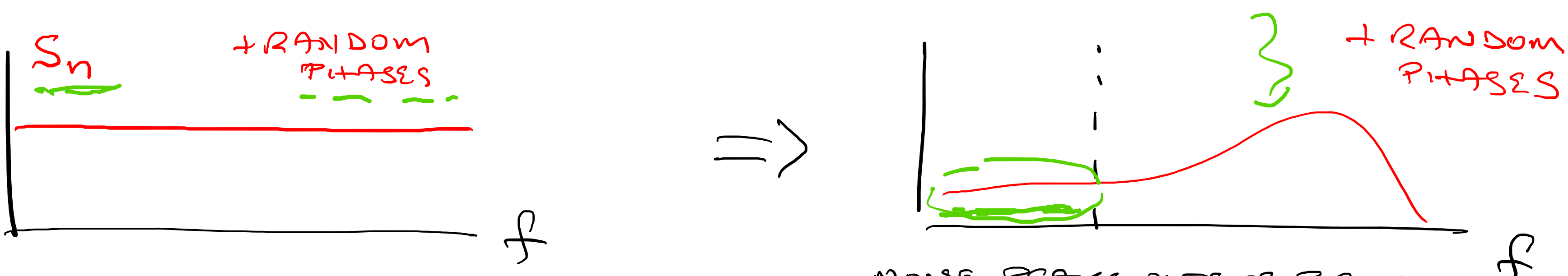
FOURIER DOMAIN:





ALTERNATIVELY, SHAPED NOISE

INSTEAD OF



NOISE PEAKS OUTSIDE BAND
OF ANALOG SIGNAL

ADC / DAC HARDWARE

→ INTERFACING TO EXPERIMENT

→ WAVEFORM GENERATION

→ DATA ACQUISITION + STORAGE

→ A/D FOR NOISE-FREE X'MISSION OF ANALOG INFORMATION

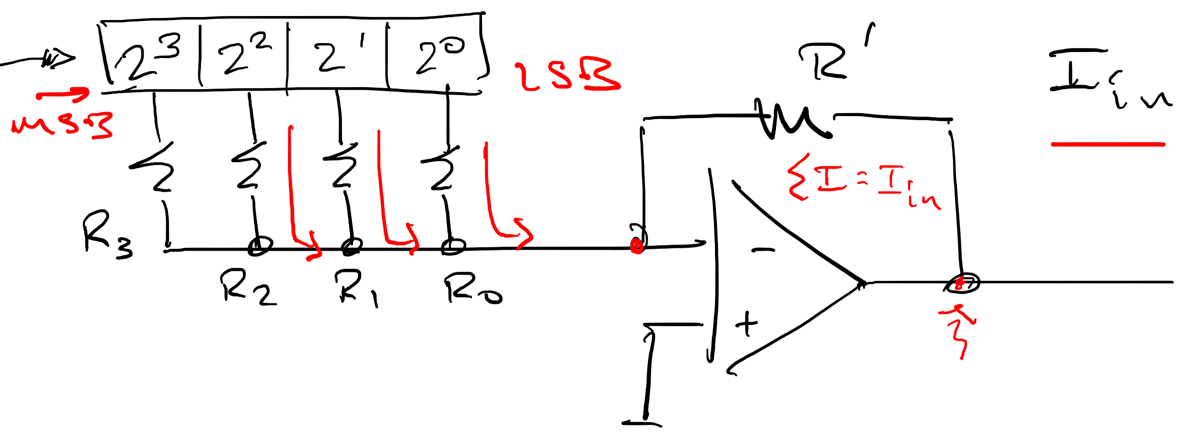
DAC → CONVERT BINARY INPUT CODE TO ANALOG OUTPUT

e.g. 4 BITS

SHIFT REGISTER

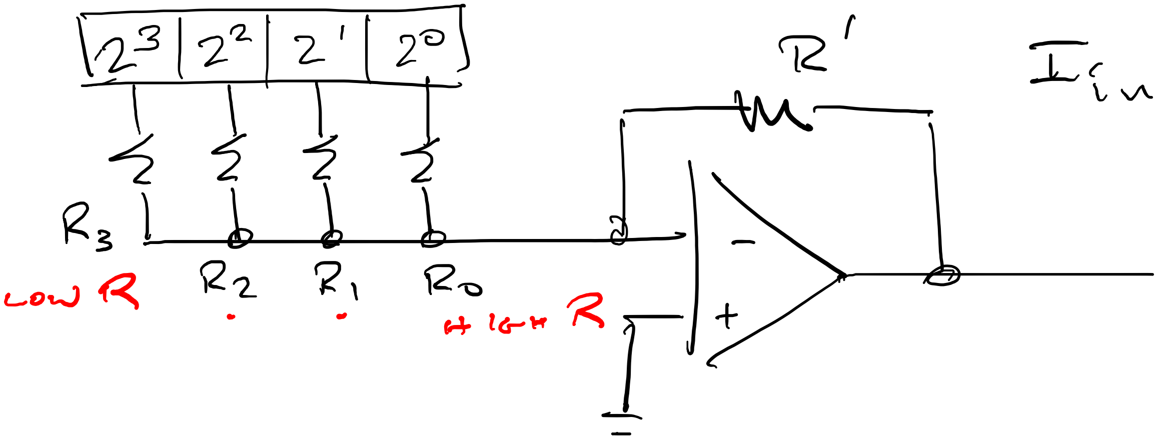
"HIGH" = +1

"LOW" = 0



$$I_{in} = \frac{V_3}{R_3} + \frac{V_2}{R_2} + \frac{V_1}{R_1} + \frac{V_0}{R_0}$$

$$V_x = 0, 1$$



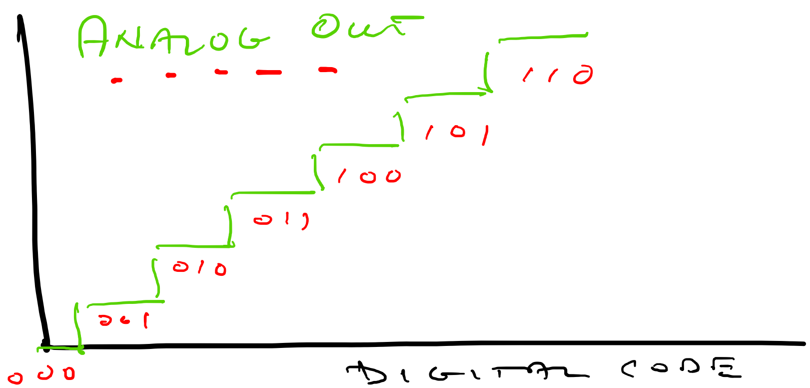
$$I_{in} = \frac{V_3}{R_3} + \frac{V_2}{R_2} + \frac{V_1}{R_1} + \frac{V_0}{R_0}$$

$$V_x = 0, 1$$

CHOOSE $R_0 = 2R_1 \; \ldots \; R_x = 2R_{x+1}$

→ RESISTORS INVERSELY PROPORTIONAL TO BINARY WEIGHTS

$$R_0 = R \; ; \; R_1 = R/2 \; ; \; R_2 = R/4 \; \ldots \; R_x = R/2^x$$



$$|V_{out,max}| = (2^n - 1)\, R'/R$$

$$\text{SMALLEST INCREMENT} = \frac{|V_{out,max}|}{2^n - 1}$$

e.g. 8-BIT DAC, 10 V FS RANGE

$$\Delta V = 10V/255 \simeq 39\ mV$$

CONCLUSIONS:

① PRECISION LIMITED BY LSB

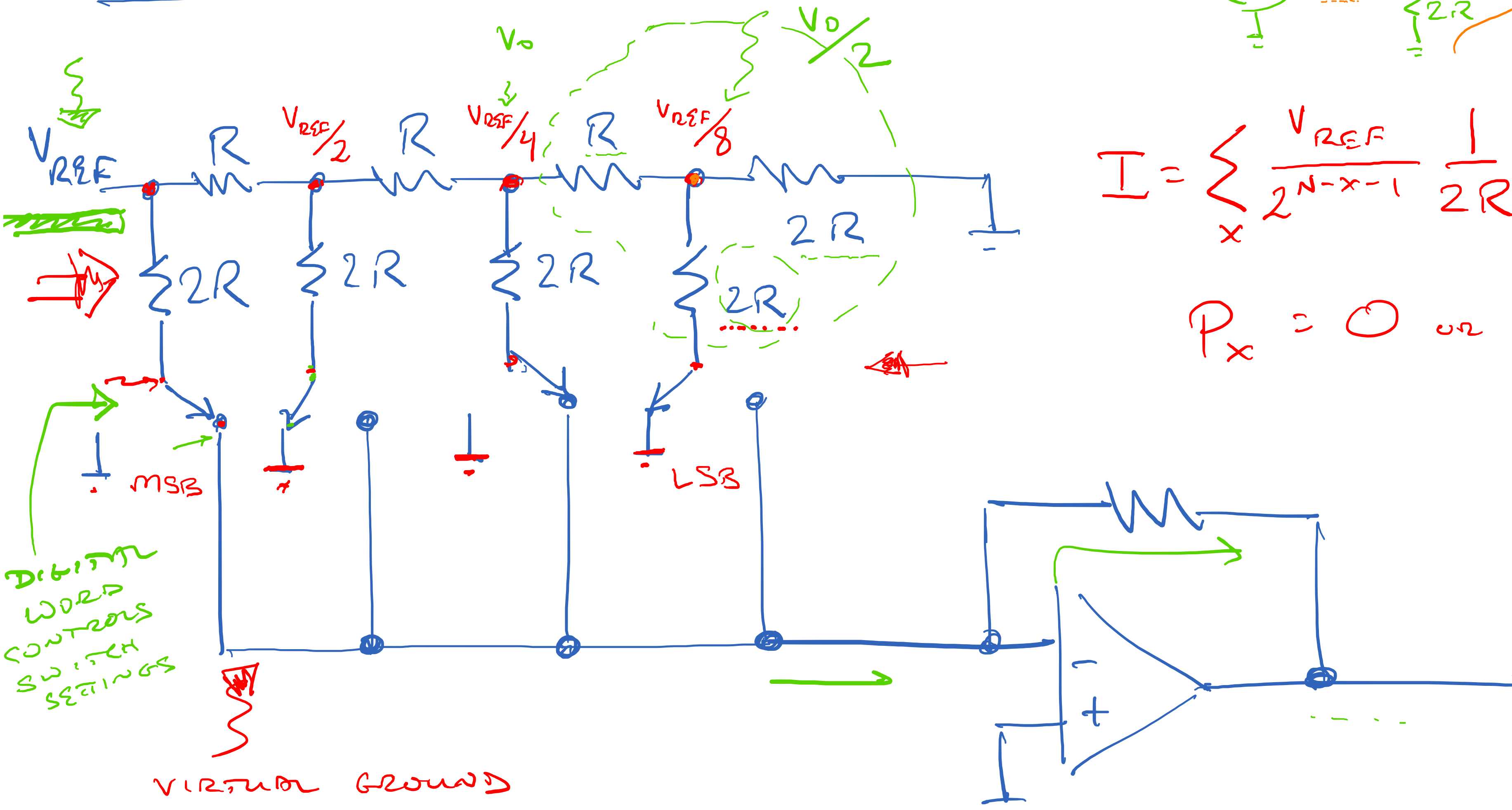
② PRECISION ALSO DETERMINED BY PRECISION → OF R_n RESISTOR [SMALLEST R_x, CORRESPONDING TO MSB]

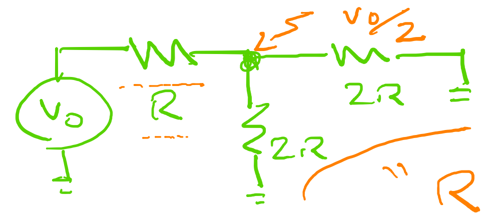
→ FOR MORE SIGNIFICANT BITS, NEED VERY

→ HIGH PRECISION FOR SMALL RESISTOR VALUES

e.g.: 10 BITS → NEED 1:1000 PRECISION ON R FOR MSB ⇒ DIFFICULT IN PRACTICE

BETTER: R / 2R LADDER





$$I = \sum_x \frac{V_{REF}}{2^{N-x-1}} \frac{1}{2R} P_x ;$$

$P_x = 0$ or 1

AD CONVERSION

e.g. AUDIO ~20 Hz → 20 kHz

CD: 44 kHz SAMPLING RATE

2 × 16 BIT CHANNELS [~ 1:65K RESOLUTION]

MP3: 16 kHz SAMPLING RATE

8 BIT CHANNEL 1:256 RESOLUTION

TRADEOFF IN ADC: SPEED vs RESOLUTION

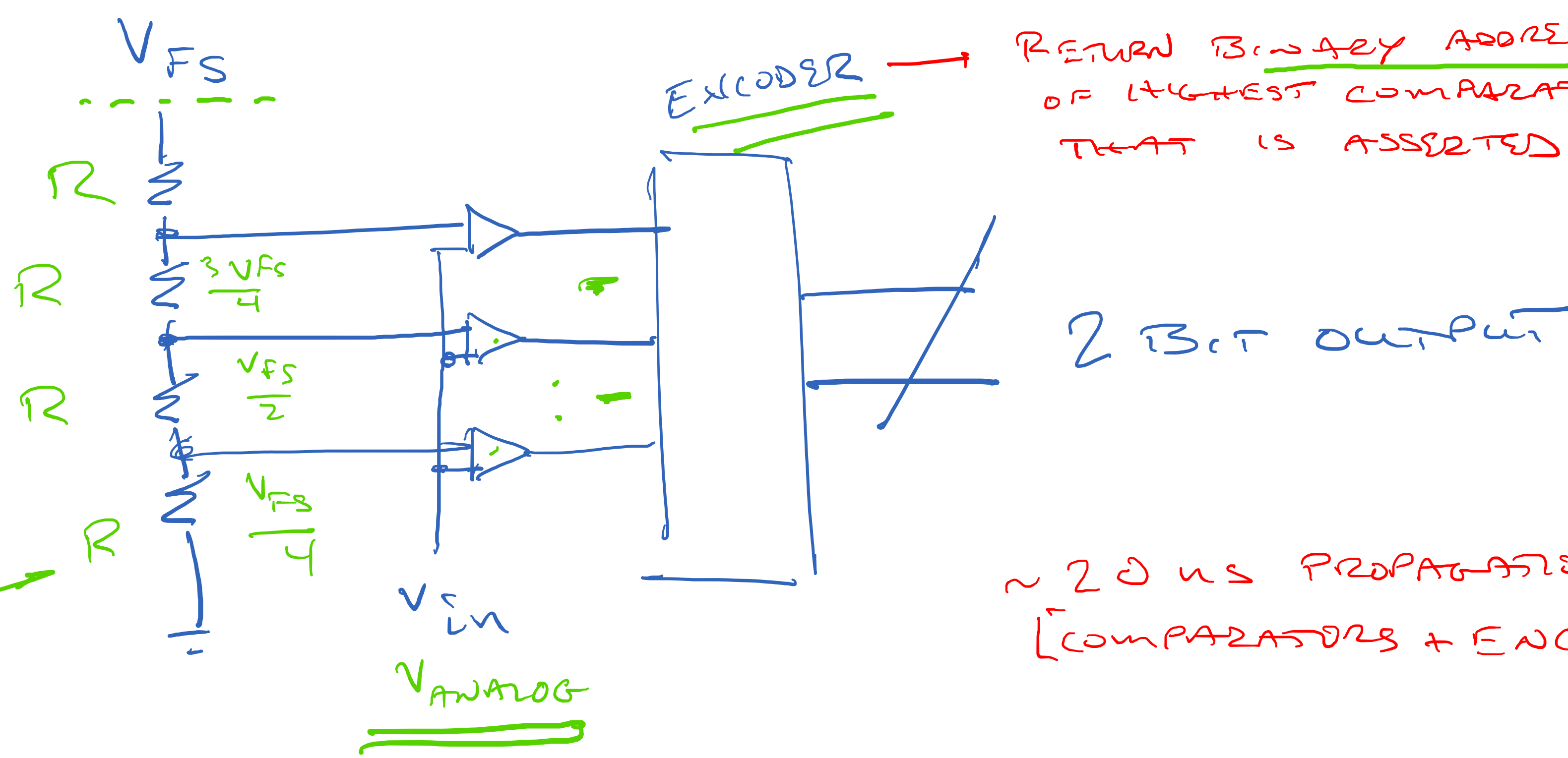
BANDWIDTH DYNAMIC RANGE

OTHER CONSIDERATIONS: LINEARITY, ABSOLUTE ACCURACY, ETC.

FAST : PARALLEL ENCODING / FLASH ADC

n BITS : DIVIDE V_{FS} INTO 2^n EQUAL INTERVALS

[REQ. $2^n - 1$ COMPARATORS]

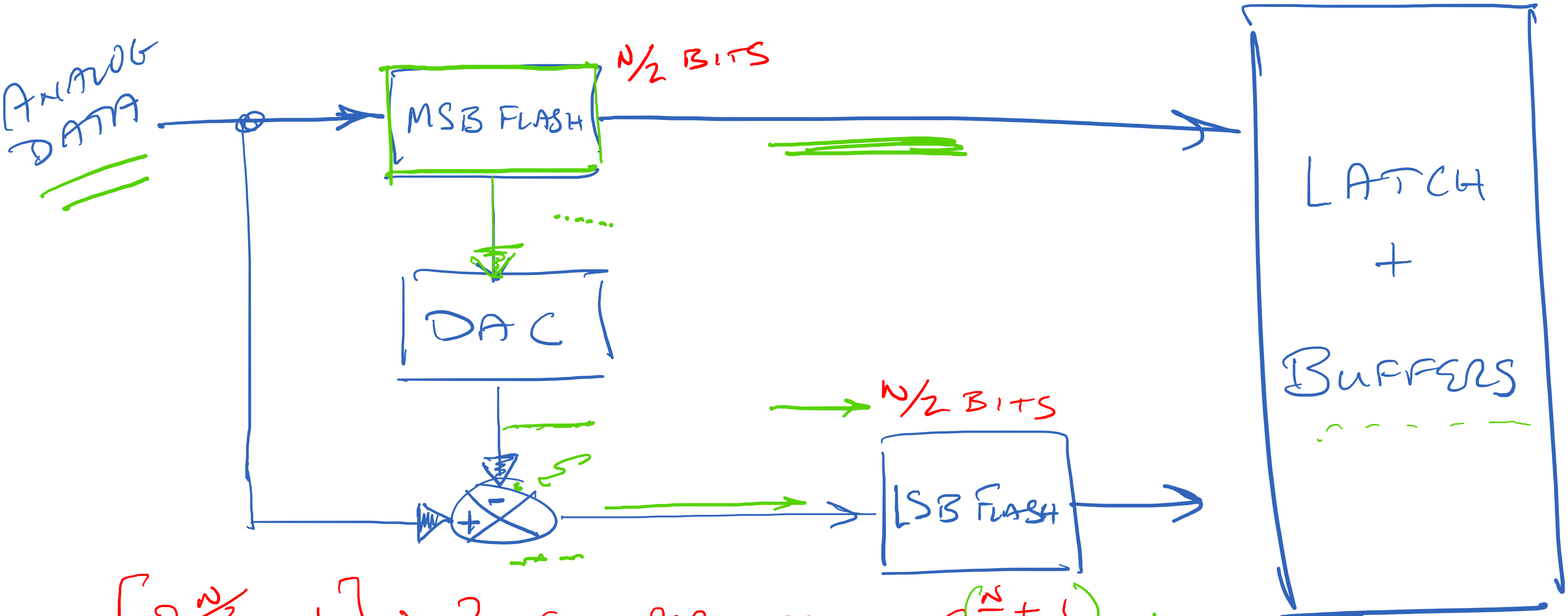


RETURN BINARY ADDRESS OF HIGHEST COMPARATOR THAT IS ASSERTED

~ 20 ns PROPAGATION DELAY [COMPARATORS + ENCODER]

HIGHER RESOLUTION, SLIGHTLY SLOWER:

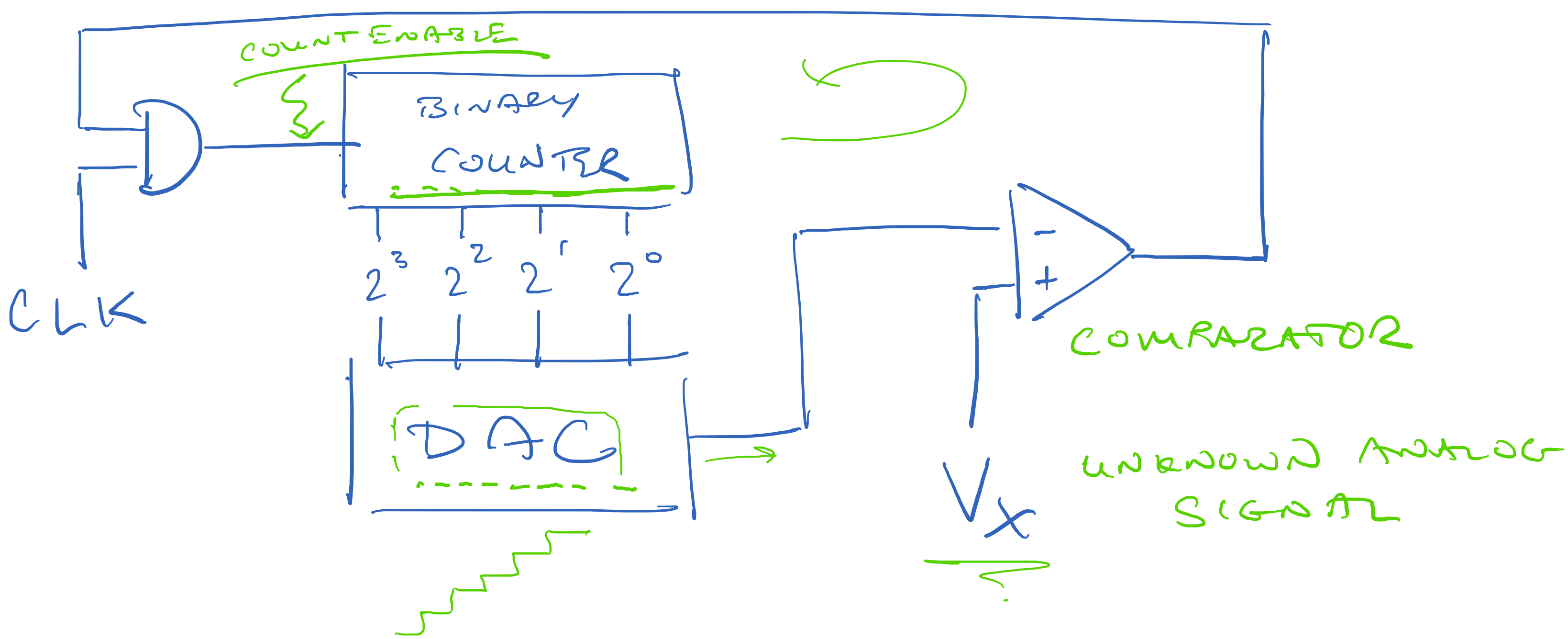
"1/2 FLASH"



$\left[2^{N/2}-1\right] \times 2$ COMPARATORS $\approx 2^{\left(\frac{N}{2}+1\right)}$

$\approx$ 1/2 AS FAST AS FULL FLASH

IN LAB: SIMPLE ADC BASED ON
BINARY COUNTER + DAC IN FEEDBACK LOOP



VARIANT OF SINGLE-SLOPE ADC