Physics 623: HW 10

Some practice with the convolution theorem.

reference: The Fourier transform cheatsheet contains all the transform pairs you need *and* a diagram showing the convolution theorem. It's available on the course website if you didn't pick up a copy.

1. If you have a voltage signal *V(t)* and would like to know its frequency spectrum *V(f),* you will probably want to use a computer to approximate the Fourier transform integral. But the integral requires a continuous function in time, and you can only digitize *V(t)* at some interval Δt and put this discrete set of numbers into the computer and do the Fourier transform on them.

You could imagine generating the discrete set of points by taking the product of *V(t)* with a "picket fence" of delta functions spaced Δt apart. Use the convolution theorem to prove the Nyquist sampling theorem: If *V(f)* is identically zero for all $f > f_{NYQUIST}$, where $f_{NYQUIST} \equiv 1/2\Delta t$, then the F.T. of the sampled waveform will be identical to *V(f)* for $f < f_{NYQUIST}$.

2. Another problem with your computed approximation for V(f) is that the Fourier integral goes from $-\infty < t < \infty$, and you probably don't want to take data for that long. Suppose you take data from $t = -T/2$ to $t = +T/2$. Use the convolution theorem to describe the effect on an arbitrary spectrum. Sketch both the true *V(f)* and the *V(f)* obtained from the computer for $V(t) = \cos(2\pi 9t) + \cos(2\pi 11t)$. What is the minimum sampling rate required to avoid aliasing?

3. For fun: Show that the Heisenberg uncertainty principle, $\Delta p \Delta x \geq \frac{\hbar}{2}$, is an exact equality if the probability distribution for the position, x, is gaussian. Δp and Δx are the r.m.s. uncertainties in position and momentum. Note that the r.m.s. deviation of a gaussian $e^{\frac{\Delta x^2}{2\sigma^2}}$ is σ, and the square of a gaussian is another gaussian with r.m.s. deviation smaller by $1/\sqrt{2}$.

For non-physicists: The momentum $p = \frac{h}{\lambda} \equiv hw$, where w is the wavenumber (cycles/meter). The probability distribution in position is $|\Psi(x)|^2$ and the probability distribution in w is $|\Psi(w)|^2$, where $\Psi(x)$ and $\Psi(w)$ are a Fourier transform pair. For a pure momentum state, $\Psi(x) = \Psi_0 e^{i2\pi w_0 x}$. To localize this with a probability distribution in x, we can replace Ψ_0 by a gaussian of some width to make a "wave packet". (Note that $\hbar \equiv h/2\pi$.)

4. Use the convolution theorem to prove the trigonometric identity:

$$\cos(\omega_0 t) \bullet \cos(\omega_1 t) = \frac{1}{2}\left(\cos((\omega_0 + \omega_1)t) + \cos((\omega_0 - \omega_1)t)\right)$$

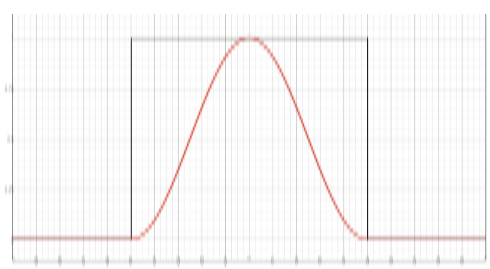
(This is easier to sketch and think about if you make $|\omega_0 - \omega_1|/\omega_0 << 1$.)

The phase detector for the lab we did is effectively a multiplier that takes advantage of this to convert two frequencies into their sum and difference. The technique, called heterodyne, is also widely used in radio receivers and other instruments.

5. Use the convolution theorem to find the apparent frequency spectrum derived from a 1-second observation of the sum of two cosine waves, one at f = 9 Hz and one at f = 11 Hz. Both have amplitudes of 1 V peak. Sketch the spectrum that would be obtained. (Note that you can mathematically reproduce a one-second observation by multiplying the infinite time sequence by a rectangular function that is 1.0 between t = –0.5 s and t = +0.5 s and zero elsewhere.)

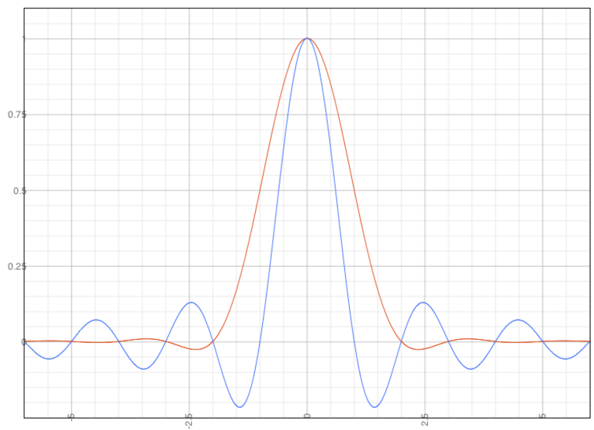
6. The sinc function that smears the spectrum observed for a finite length of time is shown in blue at the left below (compare problem 3). The oscillations can create confusing features in a spectrum that has both strong and weak sharp lines in it. The problem can be alleviated by multiplying the sampled time function by another "window" function that falls off more gently before doing the Fourier transform. One common function that is used is called a "Hanning window". If the time sequence is observed for a time *T*, the Hanning window function is:

$$\begin{array}{ll} \frac{1}{2}\left(1+\cos\frac{2\pi t}{T}\right) & \text{for } -\frac{T}{2} < t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{array}$$



Hanning window (red) — time

This goes smoothly to zero as the ends of the observing interval are approached. The new smearing function will be the Fourier transform of this Hanning window. You can use the convolution theorem again to find this transform without doing any integrals (some nasty addition required) if you construct the Hanning window as a continuous 1+cos multiplied by a rectangle function of length *T*. (Since FTs are linear, the FT of a sum is just the sum of the FTs. You need to know that the FT of a constant is a delta function at zero frequency to do the "1" part. Both terms have unit amplitude, but the cosine amplitude is split half and half between positive and negative frequencies.) The result is the red curve on the left, although this won't be obvious unless you plot your solution. You can do this, or just give the formula and plug in a couple of key points to check against red plot on the left below.) Applying this type of window to a data sample is called "apodization", or "removing the feet". Note the tradeoff — the Hanning window greatly reduces the extraneous features far from the main response, but there is significant loss of resolution in the main peak (it is broader). The squares of these smearing functions are shown on the right in blue and red and the absolute values in green and orange. These are what would appear in power spectra (V^2) and $V_{\text{r.m.s.}}$ spectra respectively.



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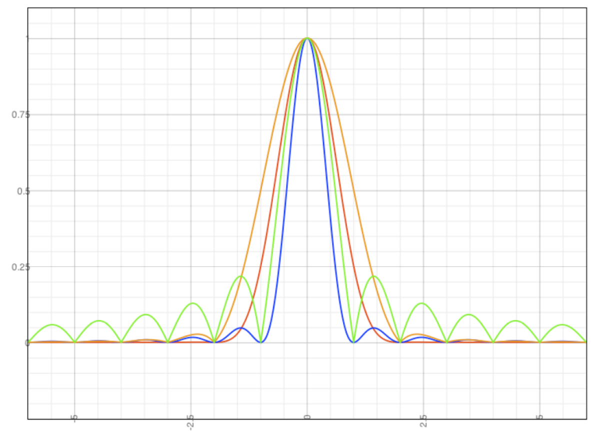

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