(b) $(x^{-2}y^{-2} + xy^{-3})dx + N(x,y)dy = 0.$

17. Consider the differential equation

$$(4x + 3y^2)dx + 2xydy = 0.$$

(a) Show that this equation is not exact.
(b) Find an integrating factor of the form x^n, where n is a positive integer.
(c) Multiply the given equation through by the integrating factor found in (b) and solve the resulting exact equation.

18. Consider a differential equation of the form

$$[y + xf(x^2 + y^2)]dx + [yf(x^2 + y^2) - x]dy = 0.$$

(a) Show that an equation of this form is not exact.
(b) Show that $1/(x^2 + y^2)$ is an integrating factor of an equation of this form.

19. Use the result of Exercise 18(b) to solve the equation

$$[y + x(x^2 + y^2)^2]dx + [y(x^2 + y^2)^2 - x]dy = 0.$$

2.2 Separable Equations and Equations Reducible to This Form

A. Separable Equations

DEFINITION. An equation of the form

$$F(x)G(y)dx + f(x)g(y)dy = 0 \tag{2.17}$$

is called an *equation with variables separable* or simply a *separable equation.*

For example, the equation $(x^3 + x^2)ydx + x^2(y^3 + 2y)dy = 0$ is a separable equation.

In general the separable equation (2.17) is not exact, but it possesses an obvious integrating factor, namely $1/f(x)G(y)$. For if we multiply Equation (2.17) by this expression, we separate the variables, reducing (2.17) to the equivalent equation

$$\frac{F(x)}{f(x)}dx + \frac{g(y)}{G(y)}dy = 0. \tag{2.18}$$

This equation is exact, since

$$\frac{\partial}{\partial y}\left[\frac{F(x)}{f(x)}\right] = 0 = \frac{\partial}{\partial x}\left[\frac{g(y)}{G(y)}\right].$$

Denoting $F(x)/f(x)$ by $M(x)$ and $g(y)/G(y)$ by $N(y)$, Equation (2.18) takes the form $M(x)dx + N(y)dy = 0$. Since M is a function of x only and N is a function of y only, we see at once that the solution is

$$\int M(x)dx + \int N(y)dy = c, \tag{2.19}$$

where c is an arbitrary constant. Thus the problem of solving the separable equation (2.17) has reduced to that of performing the integrations indicated in Equation (2.19). It is in this sense that separable equations are the simplest first-order differential equations.

Example 2.8. $(x^3 + x^2)ydx + x^2(y^3 + 2y)dy = 0$. Separating the variables by dividing by x^2y we obtain

$$\frac{x^3 + x^2}{x^2}dx + \frac{y^3 + 2y}{y}dy = 0$$

or

$$(x + 1)dx + (y^2 + 2)dy = 0.$$

From this we have

$$\int (x + 1)dx + \int (y^2 + 2)dy = c$$

or

$$\frac{x^2}{2} + x + \frac{y^3}{3} + 2y = c.$$

Observe that in separating the variables we divided by x^2y. We did this under the tacit assumption that neither x nor y is zero. Since our purpose here is to gain facility in solving separable equations, we shall always make the assumption that any factors by which we divide are not zero.

Example 2.9. Solve the initial-value problem which consists of the differential equation

(2.20) $$x\sin y dx + (x^2 + 1) \cos y dy = 0$$

and the initial condition

(2.21) $$y(1) = \frac{\pi}{2}$$

We first obtain the general solution of the differential equation (2.20). Separating the variables by dividing by $(x^2 + 1) \sin y$, we obtain

$$\frac{x}{x^2 + 1}dx + \frac{\cos y}{\sin y}dy = 0.$$

Thus

$$\int \frac{xdx}{x^2 + 1} + \int \frac{\cos y}{\sin y}dy = c_0,$$

where c_0 is an arbitrary constant.
Carrying out the integrations, we find

(2.22) $$\tfrac{1}{2} \ln (x^2 + 1) + \ln |\sin y| = c_0.$$

We could leave the general solution in this form, but we can put it in a neater form in the following way. Since each term of the left member of this equation involves the logarithm of a function, it would seem reasonable that something might be accomplished by writing the arbitrary constant c_0 in the form $\ln |c_1|$. This we do, obtaining

$$\tfrac{1}{2} \ln (x^2 + 1) + \ln |\sin y| = \ln |c_1|.$$

Multiplying by 2, we have

$$\ln (x^2 + 1) + 2 \ln |\sin y| = 2 \ln |c_1|.$$

Since

$$2 \ln |\sin y| = \ln (\sin y)^2,$$

and

$$2 \ln |c_1| = \ln c_1^2 = \ln |c|,$$

where

$$|c| = c_1^2,$$

we now have

$$\ln (x^2 + 1) + \ln \sin^2 y = \ln |c|.$$

Since $\ln A + \ln B = \ln AB$, this equation may be written

$$\ln (x^2 + 1) \sin^2 y = \ln |c|.$$

From this we have at once

$$(x^2 + 1) \sin^2 y = c. \qquad (2.23)$$

Clearly (2.23) is of a neater form than (2.22).

We now apply the initial condition (2.21) to the general solution (2.23). We have

$$(1^2 + 1) \sin^2 \frac{\pi}{2} = c$$

and so $c = 2$. Therefore the solution of the initial-value problem under consideration is

$$(x^2 + 1) \sin^2 y = 2.$$

B. Homogeneous Equations

We now consider a class of differential equations which can be reduced to separable equations by a change of variables.

DEFINITION. The first-order differential equation $Mdx + Ndy = 0$ is said to be *homogeneous* if, when written in the form $\frac{dy}{dx} = f(x,y)$, there exists g such that $f(x,y)$ can be expressed in the form $g\left(\frac{y}{x}\right)$.

Example 2.10. The differential equation $(x^2 - 3y^2)dx + 2xydy = 0$ is homogeneous. To see this, we first write this equation in the form

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}.$$

Now observing that

$$\frac{3y^2 - x^2}{2xy} = \frac{3y}{2x} - \frac{x}{2y} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left[\frac{1}{(y/x)}\right],$$

we see that the differential equation under consideration may be written as

$$\frac{dy}{dx} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left[\frac{1}{(y/x)}\right],$$

in which the right member is of the form $g\left(\frac{y}{x}\right)$ for a certain g.

Example 2.11. The equation

$$(y + \sqrt{x^2 + y^2})dx - xdy = 0$$

is homogeneous. When written in the form

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x},$$

the right member may be expressed as

$$\frac{y}{x} \pm \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2}}$$

or $\frac{y}{x} \pm \sqrt{1 + \left(\frac{y}{x}\right)^2}$, depending on the sign of x. This is obviously of the form $g\left(\frac{y}{x}\right)$.

Before proceeding to the actual solution of homogeneous equations we shall consider a slightly different procedure for recognizing such equations. A function F is called *homogeneous of degree n* if $F(tx,ty) = t^nF(x,y)$. This means that if tx and ty are substituted for x and y, respectively, in $F(x,y)$, and if t^n is then factored out, the other factor which remains is the original expression $F(x,y)$ itself. For example, the function F given by $F(x,y) = x^2 + y^2$ is homogeneous of degree 2, since $F(tx,ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2F(x,y)$.

Now suppose the functions M and N in the differential equation $Mdx + Ndy = 0$ are both homogeneous of the *same* degree n. Then since $M(tx,ty) = t^nM(x,y)$, we have $M\left(1,\frac{y}{x}\right) = M\left(\frac{1}{x}\cdot x, \frac{1}{x}\cdot y\right) = \left(\frac{1}{x}\right)^n M(x,y)$; and, in like manner, $N\left(1,\frac{y}{x}\right) = \left(\frac{1}{x}\right)^n N(x,y)$.

Now writing the differential equation $Mdx + Ndy = 0$ in the form

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)},$$

we find

$$\frac{dy}{dx} = -\frac{\left(\frac{1}{x}\right)^{-n} M\left(1,\frac{y}{x}\right)}{\left(\frac{1}{x}\right)^{-n} N\left(1,\frac{y}{x}\right)} = -\frac{M\left(1,\frac{y}{x}\right)}{N\left(1,\frac{y}{x}\right)}.$$

Clearly the expression on the right is of the form $g(y/x)$, and so the equation $Mdx + Ndy = 0$ is homogeneous in the sense of the original definition of homogeneity. Thus we conclude that if M and N in $Mdx + Ndy = 0$ are both homogeneous functions of the same degree n, then the differential equation is a homogeneous differential equation.

Let us now look back at Examples 2.10 and 2.11 in this light. In Example 2.10, $M = x^2 - 3y^2$ and $N = 2xy$. Both M and N are homogeneous of degree 2. Thus we know at once that the equation $(x^2 - 3y^2)dx + 2xydy = 0$ is a homogeneous equation. In Example 2.11, $M = y + \sqrt{x^2 + y^2}$ and $N = -x$. Clearly N is homogeneous of degree 1.

Since

$$M(tx,ty) = ty + \sqrt{(tx)^2 + (ty)^2} = t(y + \sqrt{x^2 + y^2}) = t^1 M(x,y),$$

we see that M is also homogeneous of degree 1. Thus we conclude that the equation

$$(y + \sqrt{x^2 + y^2})dx - xdy = 0$$

is indeed homogeneous.

We now show that every homogeneous equation can be reduced to a separable equation by proving the following theorem.

THEOREM 2.3. If

(2.24) $$Mdx + Ndy = 0$$

is a homogeneous equation, then the change of variables $y = vx$ transforms (2.24) into a separable equation in the variables v and x.

Proof. Since $Mdx + Ndy = 0$ is homogeneous, it may be written in the form

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right).$$

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

and (2.24) becomes

$$v + x\frac{dv}{dx} = g(v)$$

or

$$[v - g(v)]dx + xdv = 0.$$

This equation is separable. Separating the variables we obtain

(2.25) $$\frac{dv}{v - g(v)} + \frac{dx}{x} = 0.$$

Q.E.D.

Thus to solve a homogeneous differential equation of the form (2.24), we let $y = vx$ and transform the homogeneous equation into a separable equation of the form (2.25). From this, we have

$$\int \frac{dv}{v - g(v)} + \int \frac{dx}{x} = c,$$

where c is an arbitrary constant. Letting $F(v)$ denote $\int \frac{dv}{v - g(v)}$ and returning to the original dependent variable y, the solution takes the form

$$F\left(\frac{y}{x}\right) + \ln|x| = c.$$

Example 2.12. Solve the equation

$$(x^2 - 3y^2)dx + 2xydy = 0.$$

We have already observed that this equation is homogeneous. Writing it in the form

$$\frac{dy}{dx} = -\frac{x}{2y} + \frac{3y}{2x}$$

and letting $y = vx$, we obtain

$$v + x\frac{dv}{dx} = -\frac{1}{2v} + \frac{3v}{2},$$

or

$$x\frac{dv}{dx} = -\frac{1}{2v} + \frac{v}{2},$$

or, finally,

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v}.$$

This equation is separable. Separating the variables, we obtain

$$\frac{2vdv}{v^2 - 1} = \frac{dx}{x}.$$

Integrating, we find

$$\ln|v^2 - 1| = \ln|x| + \ln|c|,$$

and hence

$$v^2 - 1 = cx.$$

Now, replacing v by $\frac{y}{x}$ we obtain the solution in the form

$$\frac{y^2}{x^2} - 1 = cx$$

or

$$y^2 - x^2 = cx^3.$$

Example 2.13. Solve the initial-value problem

$$(y + \sqrt{x^2 + y^2})dx - xdy = 0,$$
$$y(1) = 0.$$

We have seen that the differential equation is homogeneous. As before, we write it in the form

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}.$$

Since the initial x value is 1, we take $x = \sqrt{x^2}$ and obtain

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + (y/x)^2}.$$

We let $y = vx$ and obtain

$$v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}$$

or

$$x\frac{dv}{dx} = \sqrt{1 + v^2}.$$

Separating variables, we find

$$\frac{dv}{\sqrt{v^2 + 1}} = \frac{dx}{x}.$$

Using tables, we perform the required integrations to obtain

$$\ln|v + \sqrt{v^2 + 1}| = \ln|x| + \ln|c|,$$

or

$$v + \sqrt{v^2 + 1} = cx.$$

Now replacing v by $\frac{y}{x}$, we obtain the general solution of the differential equation in the form

$$\frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 1} = cx$$

or

$$y + \sqrt{x^2 + y^2} = cx^2.$$

The initial condition requires that $y = 0$ when $x = 1$. This gives $c = 1$ and hence

$$y + \sqrt{x^2 + y^2} = x^2,$$

from which it follows that

$$y = \tfrac{1}{2}(x^2 - 1).$$

Exercises

Solve each of the differential equations in Exercises 1 through 14.

1. $4xydx + (x^2 + 1)dy = 0.$
2. $(xy + 2x + y + 2)dx + (x^2 + 2x)dy = 0.$
3. $2r(s^2 + 1)dr + (r^4 + 1)ds = 0.$
4. $\csc y dx + \sec x dy = 0.$
5. $\tan\theta dr + 2r d\theta = 0.$
6. $(e^v + 1)\cos u\, du + e^v(\sin u + 1)dv = 0.$
7. $(x + 4)(y^2 + 1)dx + y(x^2 + 3x + 2)dy = 0.$
8. $(x + y)dx - xdy = 0.$
9. $(2xy + 3y^2)dx - (2xy + x^2)dy = 0.$
10. $v^3du + (u^3 - uv^2)dv = 0.$
11. $\left(x\tan\frac{y}{x} + y\right)dx - xdy = 0.$
12. $(2s^2 + 2st + t^2)ds + (s^2 + 2st - t^2)dt = 0.$