

HW 9 Solns

1) BD 7.4

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(r_{12}) \quad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

$$[p_1 + p_2, H] = [p_1 + p_2, V] = p_1 V + p_2 V$$

$$(\text{Since } [p, V]\psi = pV\psi - Vp\psi = (pV)\psi + Vp\psi - Vp\psi = (pV)\psi)$$

$$p_1 V = -i\hbar \nabla_1 V = -i\hbar (\nabla_{12} V) \left(\frac{\partial \vec{r}_{12}}{\partial \vec{r}_1} \right) = -i\hbar \nabla_{12} V$$

$$p_2 V = -i\hbar (\nabla_{12} V) \frac{\partial \vec{r}_{12}}{\partial \vec{r}_2} = i\hbar \nabla_{12} V$$

$$\therefore [p_1 + p_2, V] = 0$$

System

$$H = \sum_i \frac{p_i^2}{2m} + \sum_{i < j} V(r_{ij})$$

$$[P, \sum_{i < j} V(r_{ij})] = \sum_{i < j} [p_i + p_j, V(r_{ij})] = 0$$

7.9) $H = p^2/2m + \lambda x^n$

$$[H, xp] = \frac{1}{2m} [p^2, xp] + \lambda [x^n, xp]$$

$$= [H, x]p + x[H, p]$$

$$= \left[\frac{p^2}{2m}, x \right] p + x \lambda [x^n, p]$$

$$= -\frac{1}{2m} ([x, p]p + p[x, p]) + \lambda x \sum_{s=0}^{n-1} x^s [p, x] x^{n-s-1}$$

$$= -\frac{1}{2m} (2i\hbar)p^2 + i\hbar \lambda x \sum_{s=0}^{n-1} x^{n-1-s}$$

$$= -i\hbar \frac{p^2}{m} + i\hbar n \lambda x^n = i\hbar \left(-\frac{p^2}{m} + n \lambda x^n \right)$$

$$7.9b) \quad \langle m | [H, x p] | m \rangle = \langle m | H x p - x p H | m \rangle \\ = E_m \langle m | x p - x p | m \rangle = 0$$

$$\text{but } \langle m | [H, x p] | m \rangle = i \hbar \langle m | \frac{p^2}{m} + n V | m \rangle = 0 \\ \Rightarrow -2 \langle T \rangle + n \langle V \rangle = 0 \quad \checkmark$$

$$\text{SHO. } \langle T \rangle = \frac{\hbar \omega}{2} \langle p^2 \rangle \\ = \frac{\hbar \omega}{2} \left\langle \left(\frac{a - a^\dagger}{\sqrt{2}i} \right)^2 \right\rangle \\ = \frac{\hbar \omega}{2} \left(-\frac{1}{2} \right) \langle n | a^2 - a a^\dagger - a^\dagger a + a^{\dagger 2} | n \rangle \\ = + \frac{\hbar \omega}{4} (n+1 + n) = \frac{\hbar \omega}{4} (2n+1)$$

$$\langle V \rangle = \frac{\hbar \omega}{2} \langle x^2 \rangle = \frac{\hbar \omega}{2} \left\langle \left(\frac{a + a^\dagger}{\sqrt{2}} \right)^2 \right\rangle \\ = \frac{\hbar \omega}{4} (2n+1)$$

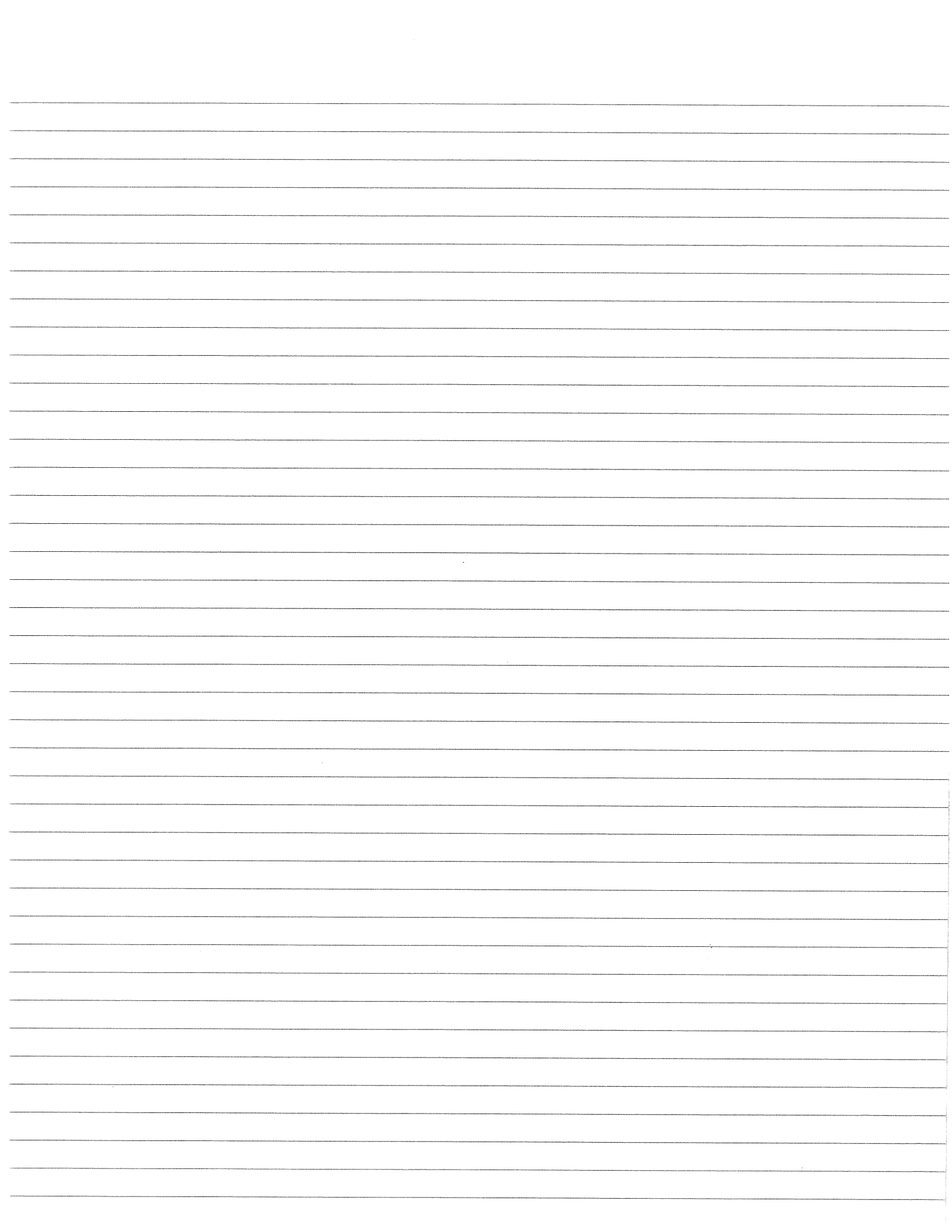
$$\langle T \rangle = \langle V \rangle = \frac{\hbar \omega}{2} (n + \frac{1}{2}) \quad \checkmark$$

$$c) \quad [H, \vec{r} \cdot \vec{p}] = \sum_i [H, r_i p_i] = \sum_i \left[\frac{p_i^2}{2m} + r_i p_i \right] + [V, \vec{r} \cdot \vec{p}]$$

$$\left[\frac{p_i^2}{2m}, r_i p_i \right] = -i \hbar \frac{p_i^2}{m}$$

$$[V, x p_x] = [V, x] p_x + x [V, p_x] = x (-p_x V) = x i \hbar (\partial_x V)$$

$$\circ \circ \quad [H, \vec{r} \cdot \vec{p}] = -i \hbar \left(\frac{p^2}{m} - \vec{r} \cdot \vec{\nabla} V \right) = -i \hbar \left(\frac{p^2}{m} - n V \right)$$



$$\infty \quad \langle [H, \vec{r} \cdot \vec{p}] \rangle = 0 = 2\langle T \rangle - n\langle V \rangle$$

d) general $2\langle T \rangle = \langle \vec{r} \cdot \vec{\nabla} V \rangle$

$$V = V(r)$$

$$2\langle T \rangle = \langle r \frac{\partial V}{\partial r} \rangle$$

7.10) $R|\xi_n\rangle = |\xi_{n+1}\rangle$

$$R^6|\xi_n\rangle = |\xi_n\rangle \quad \text{--- ~~WRONG~~ ---}$$

$$|k\rangle = \sum_p |\xi_p\rangle \langle \xi_p | k \rangle$$

$$R|k\rangle = \lambda_k |k\rangle$$

$$R^6|k\rangle = \lambda_k^6 |k\rangle \Rightarrow \lambda_k = e^{i k \frac{2\pi}{6}} = e^{i \pi/3 k}$$

$$\sum_p \underbrace{R|\xi_p\rangle}_{|\xi_{p+1}\rangle} \langle \xi_p | k \rangle = \lambda_k \sum_p |\xi_p\rangle \langle \xi_p | k \rangle$$

$$\infty \quad \langle \xi_{p+1} | k \rangle = \lambda_k \langle \xi_p | k \rangle$$

choose ~~$\lambda_k = 1$~~ $\langle \xi_1 | k \rangle = 1/\sqrt{6}$

$$\infty \quad \langle \xi_p | k \rangle = \frac{e^{-i \frac{\pi}{3} p k}}{\sqrt{6}}$$

$$\langle \ell | k \rangle = \sum_{pq} \frac{e^{+i \frac{\pi}{3} q \ell} e^{-i \frac{\pi}{3} p k}}{6} \langle \xi_q | \xi_p \rangle$$

$$7.10 \text{ cont}) \quad \langle \ell | k \rangle = \sum_p \frac{e^{-i\frac{\pi}{3}(k-2)p}}{6} = \delta_{\ell k}$$

$$\begin{aligned} d) \quad R^{-1} |k\rangle &= \sum_p e^{-i\frac{\pi}{3}pk} |\xi_{p-1}\rangle \\ &= \sum_p e^{i\frac{\pi}{3}k} e^{-i\frac{\pi}{3}(p-1)k} |\xi_{p-1}\rangle \\ &= e^{i\frac{\pi}{3}k} |k\rangle \end{aligned}$$

$$e) \quad [A, B] = \sum_n -A \left[|\xi_{n+1}\rangle \langle \xi_n| + |\xi_{n-1}\rangle \langle \xi_n| \right]$$

$$H = -A (R + R^\dagger)$$

$$[H, R] = -A [R^\dagger, R] = -A \sum_n (R^\dagger R - R R^\dagger) = 0$$

$\therefore |k\rangle$ are eigenstates of H

$$H |k\rangle = -A \left(e^{-i\frac{\pi}{3}k} + e^{i\frac{\pi}{3}k} \right) = -2A \cos \frac{\pi}{3} k$$

$k=1, 5 \neq 2, 4$ are degenerate

$$\begin{array}{c} \uparrow \\ E \\ \downarrow \end{array} \quad \begin{array}{ccc} & -3 & \\ 2 & - & -4 \\ 1 & - & -5 \\ & -6 & \end{array}$$

g) just replace $\frac{\pi}{3} \Rightarrow \frac{\pi}{4}$

$$E = -2A \cos \frac{\pi}{4} k, \quad k=1 \dots 8$$

$$\begin{array}{ccc} & -4 & \\ 3 & - & -5 \\ 2 & - & -6 \\ 1 & - & -7 \\ & -8 & \end{array}$$

$$g) \quad |\psi(0)\rangle = |\xi_1\rangle = \sum_k |k\rangle \langle k|\xi_1\rangle$$

$$|\psi(t)\rangle = \sum_k |k\rangle \underbrace{\langle k|\xi_1\rangle}_{e^{i\pi/4 k} / \sqrt{8}} e^{-iE_k t/\hbar}$$

$$|\langle \xi_1 | \psi(t) \rangle|^2 = \sum_k |\langle \xi_1 | k \rangle|^2 e^{-iE_k t/\hbar}$$

$$= \frac{1}{8} \left(2 \cos \frac{2At}{\hbar} + 4 \cos \frac{\sqrt{2}At}{\hbar} + 2 \right)$$

since $\frac{2A}{\hbar}$ and $\frac{\sqrt{2}A}{\hbar}$ are not

would need $\frac{2A}{\hbar} = 2\pi n$ ~~and~~ $\frac{\sqrt{2}A}{\hbar}$

$$\text{and } \frac{\sqrt{2}A}{\hbar} = 2\pi n$$

need $\frac{m}{\hbar} = \sqrt{2}$ not possible.

$$h) \quad E = -2A \cos \frac{2\pi k}{N} \quad k = 1 \dots N$$

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_p e^{-2\pi i p k / N} |\xi_p\rangle$$

continuous energy levels for $N \rightarrow \infty$

$$4) -\frac{\hbar^2}{2m} \left(\frac{d^2 f}{dp^2} + \frac{1}{p} \frac{df}{dp} \right) + \frac{\hbar^2 m^2}{2m p^2} f = E f$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$\frac{d^2 f}{dp^2} + \frac{1}{p} \frac{df}{dp} + k^2 f - \frac{m^2}{p^2} f = 0$$

Solution $f = J_m(kp)$

5) Boundary condition $ka = x_{mn} \Rightarrow k = \frac{x_{mn}}{a}$

$$E = \frac{\hbar^2}{2ma^2} x_{mn}^2 = \frac{\pi^2 \hbar^2}{2ma^2} \left(n + \frac{\mu}{2} \right)^2$$

$$\mu = n - \frac{x_{mn}}{\pi} + \frac{m}{2}$$

Tables of μ 's, E_s^0 , and E_s^1 are given in Mathematica output. Largest error is 35% for $m=5, n=1$ but rapidly goes to zero at large m .

quantum defects

$$\text{In[2]:= } \mu_{mn} = \text{Table} \left[n + \frac{m}{2} - \frac{1}{\pi} \text{BesselJZero}[m, n], \{m, 0, 5\}, \{n, 1, 15\} \right] // N$$

```
Out[2]= {{0.23452, 0.242905, 0.245433, 0.246638, 0.247341, 0.247802, 0.248126, 0.248368,
0.248554, 0.248702, 0.248823, 0.248923, 0.249007, 0.249079, 0.249142},
{0.28033, 0.266869, 0.261685, 0.258937, 0.257236, 0.256078, 0.25524, 0.254605,
0.254107, 0.253707, 0.253377, 0.253102, 0.252867, 0.252666, 0.252491},
{0.365281, 0.320708, 0.30129, 0.290302, 0.283212, 0.278251, 0.274583,
0.27176, 0.26952, 0.267699, 0.266188, 0.264916, 0.263829, 0.26289, 0.26207},
{0.469131, 0.39297, 0.357133, 0.33591, 0.321791, 0.311694, 0.304104, 0.298186,
0.29344, 0.289549, 0.286299, 0.283544, 0.281179, 0.279126, 0.277327},
{0.584556, 0.477994, 0.425079, 0.392664, 0.370581, 0.354509, 0.342261,
0.332608, 0.324798, 0.318345, 0.312923, 0.308302, 0.304316, 0.300842, 0.297786},
{0.70795, 0.5725, 0.502479, 0.458436, 0.427855, 0.405271, 0.387864, 0.374016,
0.362725, 0.353337, 0.345404, 0.33861, 0.332725, 0.327577, 0.323035}}
```

actual energy levels

$$\text{In[3]:= } es = c \left(\text{Table} \left[\left(n + \frac{m}{2} \right), \{m, 0, 5\}, \{n, 1, 15\} \right] - \mu_{mn} \right)^2$$

```
Out[3]= {{0.585959 c, 3.08738 c, 7.58764 c, 14.0877 c,
22.5878 c, 33.0878 c, 45.5878 c, 60.0878 c, 76.5878 c, 95.0878 c,
115.588 c, 138.088 c, 162.588 c, 189.088 c, 217.588 c},
{1.48759 c, 4.98687 c, 10.4867 c, 17.9866 c, 27.4866 c, 38.9866 c, 52.4865 c, 67.9865 c,
85.4865 c, 104.987 c, 126.487 c, 149.987 c, 175.487 c, 202.987 c, 232.487 c},
{2.67231 c, 7.17861 c, 13.6805 c, 22.1813 c, 32.6817 c, 45.1819 c, 59.6821 c, 76.1822 c,
94.6822 c, 115.182 c, 137.682 c, 162.182 c, 188.682 c, 217.182 c, 247.682 c},
{4.12443 c, 9.65364 c, 17.1633 c, 26.6678 c, 38.1703 c, 51.6717 c, 67.1727 c, 84.6734 c,
104.174 c, 125.674 c, 149.174 c, 174.675 c, 202.175 c, 231.675 c, 263.175 c},
{5.83437 c, 12.4045 c, 20.9299 c, 31.4422 c, 43.9492 c, 58.4535 c, 74.9564 c,
93.4585 c, 113.96 c, 136.461 c, 160.962 c, 187.463 c, 215.963 c, 246.464 c, 278.964 c},
{7.79554 c, 15.4253 c, 24.9752 c, 36.5005 c, 50.0152 c, 65.5246 c, 83.031 c, 102.536 c,
124.039 c, 147.541 c, 173.043 c, 200.545 c, 230.046 c, 261.547 c, 295.048 c}}
```

approximate energy levels


```
In[5]:= eps = c Table[ $\left(n + \frac{m}{2} - \frac{1}{4}\right)^2$ , {m, 0, 5}, {n, 1, 15}] // N
```

```
Out[5]= {{0.5625 c, 3.0625 c, 7.5625 c, 14.0625 c, 22.5625 c, 33.0625 c, 45.5625 c, 60.0625 c,
76.5625 c, 95.0625 c, 115.563 c, 138.063 c, 162.563 c, 189.063 c, 217.563 c},
{1.5625 c, 5.0625 c, 10.5625 c, 18.0625 c, 27.5625 c, 39.0625 c, 52.5625 c, 68.0625 c,
85.5625 c, 105.063 c, 126.563 c, 150.063 c, 175.563 c, 203.063 c, 232.563 c},
{3.0625 c, 7.5625 c, 14.0625 c, 22.5625 c, 33.0625 c, 45.5625 c, 60.0625 c, 76.5625 c,
95.0625 c, 115.563 c, 138.063 c, 162.563 c, 189.063 c, 217.563 c, 248.063 c},
{5.0625 c, 10.5625 c, 18.0625 c, 27.5625 c, 39.0625 c, 52.5625 c, 68.0625 c, 85.5625 c,
105.063 c, 126.563 c, 150.063 c, 175.563 c, 203.063 c, 232.563 c, 264.063 c},
{7.5625 c, 14.0625 c, 22.5625 c, 33.0625 c, 45.5625 c, 60.0625 c, 76.5625 c, 95.0625 c,
115.563 c, 138.063 c, 162.563 c, 189.063 c, 217.563 c, 248.063 c, 280.563 c},
{10.5625 c, 18.0625 c, 27.5625 c, 39.0625 c, 52.5625 c, 68.0625 c, 85.5625 c, 105.063 c,
126.563 c, 150.063 c, 175.563 c, 203.063 c, 232.563 c, 264.063 c, 297.563 c}}
```

percentage error

```
In[8]:= 100  $\frac{\text{eps} - \text{es}}{\text{es}}$  // MatrixForm
```

```
Out[8]//MatrixForm=

$$\begin{pmatrix} -4.00356 & -0.806002 & -0.331331 & -0.179065 & -0.111852 & -0.0764193 & -0.0554922 & -0.0421 \\ 5.03533 & 1.51654 & 0.722944 & 0.421901 & 0.276214 & 0.19479 & 0.144712 & 0.1117 \\ 14.6013 & 5.34776 & 2.79261 & 1.71878 & 1.16528 & 0.842353 & 0.637439 & 0.4992 \\ 22.7443 & 9.41473 & 5.23879 & 3.3549 & 2.33752 & 1.72387 & 1.32463 & 1.050 \\ 29.6198 & 13.3659 & 7.80034 & 5.1532 & 3.67085 & 2.75255 & 2.14266 & 1.716 \\ 35.4941 & 17.0969 & 10.3594 & 7.01909 & 5.09297 & 3.87314 & 3.04884 & 2.464 \end{pmatrix}$$

```