

HW 7

6.1) 

$$H = \begin{pmatrix} E_0 & -a & 0 \\ -a & E_0 & -a \\ 0 & -a & E_0 \end{pmatrix}$$

$\mathcal{M}: E = E_0 + \{-\sqrt{2}a, 0, \sqrt{2}a\}$

$$|0\rangle = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \quad |-\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \quad |+\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

b) ground state $L: 1/4 \quad C: 1/2 \quad R: 1/4$

c) $\langle 0|L\rangle = -1/\sqrt{2} \quad \langle -|L\rangle = 1/2 \quad \langle +|L\rangle = 1/2$

$\therefore P(E_0) = 1/2 \quad P(E_0 + \sqrt{2}a) = P(E_0 - \sqrt{2}a) = 1/4$

$$\langle E \rangle = E_0 + \frac{1}{2} \cdot 0 + \frac{1}{4}(\sqrt{2}a) + \frac{1}{4}(-\sqrt{2}a) = E_0$$

$$\begin{aligned} \langle E^2 \rangle &= \frac{1}{2} E_0^2 + \frac{1}{4} (E_0 + \sqrt{2}a)^2 + \frac{1}{4} (E_0 - \sqrt{2}a)^2 \\ &= E_0^2 + a^2 \end{aligned}$$

$$\Delta E = a$$

6.2) $H = \begin{pmatrix} 0 & -A & -A \\ -A & 0 & A \\ -A & -A & 0 \end{pmatrix}$

Ammonia $\begin{pmatrix} 0 & -A \\ -A & 0 \end{pmatrix} \Rightarrow E = \pm A$

quite similar

$$M: \left. \begin{aligned} \langle \phi_1 | H | \phi_1 \rangle &= -2a = \langle E \rangle \\ \langle \phi_1 | H^2 | \phi_1 \rangle &= 4a^2 \end{aligned} \right\} \Delta E = \sqrt{4a^2 - 4a^2} = 0$$

must be an eigenstate

$$\langle \phi_2 | H | \phi_2 \rangle = a \Rightarrow \Delta E = 0$$

$$\langle \phi_2 | H^2 | \phi_2 \rangle = a^2$$

$$c) E = \{-2a, a, a\}, \quad |1\rangle = \{1, 1, 1\}$$

$$|1\rangle = (1, 0, -1)$$

$$|1'\rangle = (-1, 1, 0)$$

Since $|1\rangle$ & $|1'\rangle$ both have $E=a$, any linear combination of them is also an eigenstate. The basis is not unique

d) energy splitting is $3a = 2.75 \text{ eV}$, so yellow light is absorbed. So transmitted light looks violet.

$$e) \text{ new } H = \begin{pmatrix} \Delta & -A & -A \\ -A & 0 & -A \\ -A & -A & 0 \end{pmatrix}$$

$$M: \text{ eigenvalues } a, \frac{\Delta}{2} - \frac{a}{2} \pm \frac{1}{2} \sqrt{9a^2 + 2\Delta a + \Delta^2}$$

$$\Delta \ll a, \text{ get } -\frac{a}{2} \pm \frac{3a}{2} = a, -2a \text{ as before}$$

$$\Delta \gg a, \text{ get } 0, 0, \Delta$$

$$f) h\nu_1 = a - \left(\frac{\Delta}{2} - \frac{a}{2} - \frac{1}{2} \sqrt{9a^2 + 2\Delta a + \Delta^2} \right) = \frac{3a}{2} - \frac{\Delta}{2} + \frac{1}{2} \sqrt{\dots}$$

$$\approx h\nu_2 = \sqrt{9a^2 + 2\Delta a + \Delta^2}.$$

$$h\nu_1 = \frac{3a-\Delta}{2} + \frac{h\nu_c}{2}$$

$$3a-\Delta = 2h\nu_1 - h\nu_2 = 2(2\text{eV}) - 2.75\text{eV} = 1.25\text{eV}$$

$$\Delta = 2.25\text{eV} - 1.0\text{eV} = 1.25\text{eV}$$

check $\sqrt{9a^2 + 2\Delta a + \Delta^2} = 2.75\text{eV} \checkmark$

with $\Delta = 1.0\text{eV}$, works well.

$$3) \quad H = \hbar \begin{pmatrix} 0 & \epsilon/2 \\ \epsilon/2 & 0 \end{pmatrix} \Rightarrow E = \pm \frac{\hbar\epsilon}{2} \quad \begin{aligned} |+\rangle &= (1, 1)/\sqrt{2} \\ |-\rangle &= (1, -1)/\sqrt{2} \end{aligned}$$

$$(U^\dagger H U) = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$$

$$\begin{aligned} \psi(t) &= |+\rangle \langle +|\psi_0\rangle e^{-iEt/\hbar} + |-\rangle \langle -|\psi_0\rangle e^{+iEt/\hbar} \\ &= (|+\rangle + |-\rangle e^{-iEt/\hbar} + |-\rangle - |+\rangle e^{+iEt/\hbar}) |\psi(0)\rangle \\ &= U(t) \psi(0) \end{aligned}$$

$$U = \begin{pmatrix} U_{gg} & U_{ge} \\ U_{eg} & U_{ee} \end{pmatrix}$$

$$\begin{aligned} \langle g|U|g\rangle &= \langle g|+\rangle \langle +|g\rangle e^{-iEt/\hbar} + \langle g|-\rangle \langle -|g\rangle e^{+iEt/\hbar} \\ &= \cos \epsilon t/2 \end{aligned}$$

$$\langle g|U|e\rangle = \frac{1}{2} e^{-iEt/\hbar} + \frac{1}{2} \left(\frac{-1}{\sqrt{2}}\right) e^{+iEt/\hbar} = -i \sin \epsilon t/2 = U_{eg}$$

$$U = \begin{pmatrix} \cos \epsilon t/2 & -i \sin \epsilon t/2 \\ -i \sin \epsilon t/2 & \cos \epsilon t/2 \end{pmatrix}$$

$$3) \text{ cont. } U(\pi/2) |g\rangle = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -i \end{pmatrix} = -i |e\rangle$$

$$U(2\pi) |g\rangle = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -|g\rangle$$

$$4) H_0 = \begin{pmatrix} 0 & 0 \\ 0 & \hbar(\omega_0 - \omega) \end{pmatrix}$$

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i(\omega_0 - \omega)t} \end{pmatrix}$$

$$\psi = U(\pi/2) U_0(T) U(\pi/2) |g\rangle$$

$$= \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \\ \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \downarrow \begin{pmatrix} 1/\sqrt{2} & -i(\omega_0 - \omega)T \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} - \frac{1}{2} e^{-i(\omega_0 - \omega)T} \\ -\frac{i}{2} (1 + e^{-i(\omega_0 - \omega)T}) \end{pmatrix}$$

$$|\langle e | \psi \rangle|^2 = \frac{1}{4} (2 + 2 \cos \Delta T) = \cos^2 \frac{\Delta T}{2}$$

$$5) P(T) = \frac{1}{\sqrt{2\pi\sigma_T^2}} e^{-\frac{(T-\bar{T})^2}{2\sigma_T^2}}$$

$$|\langle e | \psi \rangle|^2 = \int dT \frac{e^{-\frac{(T-\bar{T})^2}{2\sigma_T^2}}}{\sqrt{2\pi\sigma_T^2}} \cos^2 \frac{\Delta T}{2} = \frac{1}{2} \left(1 + e^{-\frac{\Delta^2 \sigma_T^2}{2}} \cos \Delta \bar{T} \right)$$

The width of the resonance around $\Delta = 0$ is about 2000 rad/sec

$$\text{or } \Delta\nu = \frac{2000 \text{ rad/sec}}{2\pi \text{ rad/cycle}} = 318 \text{ Hz}$$

So, if $\omega_0 \approx 2\pi \times 10 \text{ GHz}$, our clock would have a resolution of

$$\left(\frac{10 \text{ GHz}}{318 \text{ Hz}} \right)^{-1} = 3 \times 10^{-8}$$

■ BD 6.1

In[2]:= **{eval, kets} = Eigensystem** $\left[\mathbf{h} = \begin{pmatrix} E0 & -a & 0 \\ -a & E0 & -a \\ 0 & -a & E0 \end{pmatrix}\right]$

Out[2]:= $\left\{\left\{E0, -\sqrt{2} a + E0, \sqrt{2} a + E0\right\}, \left\{\{-1, 0, 1\}, \{1, \sqrt{2}, 1\}, \{1, -\sqrt{2}, 1\}\right\}\right\}$

In[3]:= **bras = Inverse[kets^T]**

Out[3]:= $\left\{\left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}, \left\{\frac{1}{4}, \frac{1}{2\sqrt{2}}, \frac{1}{4}\right\}, \left\{\frac{1}{4}, -\frac{1}{2\sqrt{2}}, \frac{1}{4}\right\}\right\}$

■ BD 6.2

In[4]:= **{eval, kets} = Eigensystem** $\left[\mathbf{h} = \begin{pmatrix} 0 & -a & -a \\ -a & 0 & -a \\ -a & -a & 0 \end{pmatrix}\right]$

Out[4]:= $\left\{\{-2 a, a, a\}, \left\{\{1, 1, 1\}, \{-1, 0, 1\}, \{-1, 1, 0\}\right\}\right\}$

In[5]:= **$\phi1 = \{1, 1, 1\} / \sqrt{3}$; { $\phi1.h.\phi1$, $\phi1.h.h.\phi1$ }**

Out[5]:= $\{-2 a, 4 a^2\}$

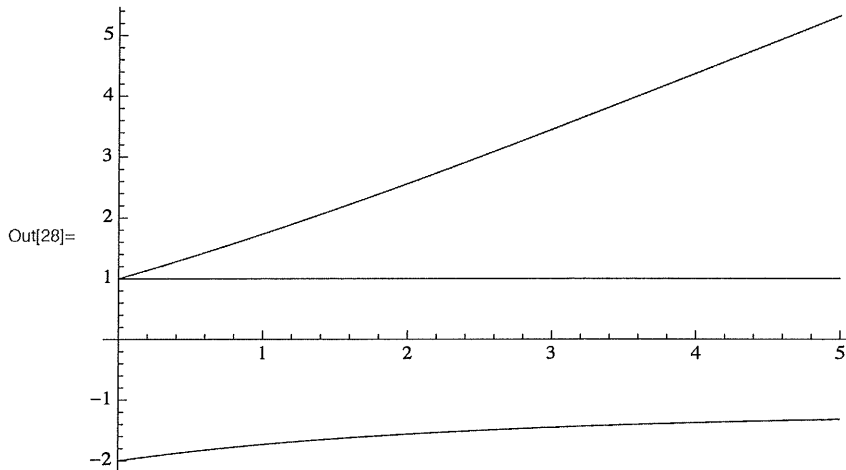
In[6]:= **$\phi2 = \{0, 1, -1\} / \sqrt{2}$; { $\phi2.h.\phi2$, $\phi2.h.h.\phi2$ }**

Out[6]:= $\{a, a^2\}$

In[27]:= **Eigenvalues** $\left[\mathbf{h} = \begin{pmatrix} \Delta & -a & -a \\ -a & 0 & -a \\ -a & -a & 0 \end{pmatrix}\right]$

Out[27]:= $\left\{a, \frac{1}{2} \left(-a + \Delta - \sqrt{9 a^2 + 2 a \Delta + \Delta^2}\right), \frac{1}{2} \left(-a + \Delta + \sqrt{9 a^2 + 2 a \Delta + \Delta^2}\right)\right\}$

In[28]:= **Plot[% /. a → 1, {Δ, 0, 5}]**



■ #5

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In[37]:= $Assumptions = {σT > 0, Δ ∈ Reals}
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Out[37]= {σT > 0, Δ ∈ Reals}
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In[60]:= Integrate[ $\frac{1}{\sqrt{2 \pi \sigma T^2}} \text{Exp}\left[\frac{-(T - \text{tbar})^2}{2 \sigma T^2}\right] \text{Cos}[\Delta T / 2]^2, \{T, -\infty, \infty\}$ ]
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Out[60]=  $\frac{1}{4} \left( 2 + e^{-\frac{1}{2} \Delta (2 i \text{tbar} + \Delta \sigma T^2)} (1 + e^{2 i \text{tbar} \Delta}) \right)$ 
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In[61]:= % /. Exp[a_ (b_ + c_)] → Exp[a b] Exp[a c]
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Out[61]=  $\frac{1}{4} \left( 2 + e^{-i \text{tbar} \Delta - \frac{\Delta^2 \sigma T^2}{2}} (1 + e^{2 i \text{tbar} \Delta}) \right)$ 
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In[65]:= res = % /. {Exp[c_ + Complex[0, a_] b_] → Exp[c] (Cos[a b] + i Sin[a b]),  
Exp[Complex[0, a_] b_] → (Cos[a b] + i Sin[a b])} // Simplify
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Out[65]=  $\frac{1}{2} \left( 1 + e^{-\frac{1}{2} \Delta^2 \sigma T^2} \text{Cos}[\text{tbar} \Delta] \right)$ 
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In[69]:= Plot[res /. {σT → tbar / 4, tbar → 100 / 50 000.},  
{Δ, -10 000, 10 000}, PlotRange → {0, All}]
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