HW 2 Solns

1) $$-\frac{\hbar^2}{2m}\nabla^2\Psi = +i\hbar\frac{\partial\Psi}{\partial t}$$

$$\Psi = \psi(r)e^{-iEt/\hbar}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

2) $$\psi = fe^{ikz}$$

$$\nabla^2\psi = e^{ikz}\nabla^2 f + 2ik\,\partial_z f e^{ikz} - k^2 f e^{ikz}$$

$$= \underbrace{-\frac{2mE}{\hbar^2}\psi = -k^2 f e^{ikz}}_{\text{using de Broglie relations}}$$

$$\therefore\quad 2ik\,\partial_z f + \underbrace{\nabla^2 f}_{\partial_x^2 f + \partial_y^2 f + \partial_z^2 f \,=\, (\nabla_\perp^2 + \partial_z^2) f} = 0$$

$$kf \gg \partial_z f \Rightarrow$$

$$\therefore\quad 2ik\,\partial_z f + \nabla_\perp^2 f = 0$$

3) Try $f = \frac{1}{q}e^{ik(x^2+y^2)/2q}$

Mathematica gives zero if $q'(z) = 1 \Rightarrow q = A + z$

$z=0 \quad \frac{ik}{2q} = -\frac{1}{w_0^2} \Rightarrow A = \frac{-ikw_0^2}{2}$

4) $q = z - \frac{ikw_0^2}{2}$ Mathematica $\Rightarrow w = w_0\sqrt{1 + \frac{4z^2}{k^2 w_0^4}}$

4 cont.) rewrite $\frac{W}{W_0} = \sqrt{1 + \left(\frac{z}{\left(\frac{kW_0^2}{2}\right)}\right)^2}$

so beam shape depends only on W_0 & k

via $z_R = \frac{kW_0^2}{2}$

Plot shown on Mathmatica pg.

5) BD 2.1 $\left(\frac{1}{c^2}\partial_t^2 - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi = 0$

$\psi \sim e^{i\vec{k}\cdot\vec{r} - i\omega t}$

$$-\frac{\omega^2}{c^2} + k^2 + \frac{m^2c^2}{\hbar^2} = 0$$

$$\omega^2 = \frac{m^2c^4}{\hbar^2} + k^2c^2$$

only ~~k > mc/ħ can propagate w/ damping~~

a frequencies $\omega > mc^2/\hbar$

b) $E = \hbar\omega = \sqrt{m^2c^4 + p^2c^2}$

c) $v_g = \frac{d\omega}{dk} = \frac{1}{2}\cdot\frac{2kc^2}{\omega} = \frac{kc^2}{\sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}}}$

$v_p = \omega/k = c^2/v_g$

b) 2.2.a) $\frac{d\langle x^2\rangle}{dt} = \int dx\, x^2(\psi^* d_t\psi + \psi d_t\psi^*)$

$= \frac{-\hbar}{2im}\int dx\, x^2(\psi^*\psi'' - \psi\psi^{*''})$

use mathematica to integrate by parts

$= \frac{-\hbar}{2im}(-2)\int dx\, x(\psi^*\psi' - \psi\psi^{*'})$

$= \frac{i\hbar}{m}\int dx\, x(\psi\psi^{*'} - \psi^*\psi') = A(t)$

b) $\frac{dA}{dt} = \frac{i\hbar}{m}\left(\frac{-\hbar}{2im}\right)\int dx\, x\left[\psi''\psi^{*'} - \psi\psi^{*'''} + c.c.\right]$

use Mathem.

$= \frac{-\hbar^2}{2m^2}\int\left[-2\psi'\psi^{*'} + c.c.\right] = \frac{2\hbar^2}{m^2}\int\psi'\psi^{*'}dx$

c) $\frac{d}{dt}\int\psi'\psi^{*'}dx = \left(\frac{-\hbar}{2im}\right)\int dx\left(\psi'''\psi^{*'} - \psi'\psi^{*'''}\right)$

$= 0$ (Mathematica)

d) $\frac{d\langle x^2\rangle}{dt} = A \Rightarrow \langle x^2\rangle = \int_0^t A\,dt' + \langle x^2\rangle_0 = \int_0^t (Bt' + \xi_0)dt'$

$\langle x^2\rangle = \langle x^2\rangle_0 + \xi_0 t + Bt^2/2 = \langle x^2\rangle_0 + \xi_0 t + v_1^2 t^2$

e) $\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2$

$= \langle x^2\rangle_0 + \xi_0 t + v_1^2 t^2 - (\langle x\rangle_0 + v_0 t)^2$

$= \langle x^2\rangle_0 - \langle x\rangle_0^2 + (\xi_0 - 2v_0\langle x_0\rangle_0)t + (\underbrace{v_1^2 - v_0^2}_{\Delta v^2})t^2$

7) 2.3)

$$\Delta p^2 = \int dp\, |\varphi(p)|^2 (p-p_0)^2 \overset{M}{=} \frac{\sigma^2 \hbar^2}{2}$$

$$\psi = \frac{1}{(2\pi\hbar)^{1/2}} \int dp\, \varphi(p)\, e^{i\frac{px}{\hbar}} \overset{M}{=} \pi^{-1/4}\sqrt{\sigma}\; e^{-\frac{1}{2}x^2\sigma^2}$$

$$\Delta x^2 = \int |\psi|^2 x^2\, dx = \frac{1}{2\sigma^2}$$

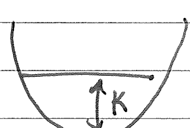
$$\therefore \quad \Delta x \Delta p = \left(\frac{1}{2\sigma^2}\right)^{1/2} \left(\frac{\sigma^2\hbar^2}{2}\right)^{1/2} = \hbar/2$$

b) $$\psi = \frac{1}{(2\pi\hbar)^{1/2}} \int dp\, \varphi(p)\, e^{i\frac{px}{\hbar}}\, e^{-i\frac{p^2}{2m\hbar}t}$$

Mathematica

$$\int |\psi|^2 x^2\, dx = \frac{1}{2\sigma^2} + \frac{\sigma^2 t^2 \hbar^2}{2m^2}$$

2.4) a) $\Delta p \Delta x \sim \hbar/2$ $\qquad \Delta p^2 = \langle p^2 \rangle = \frac{1}{2} K 2m = mK$

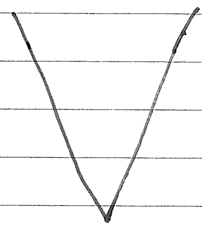


$$\frac{1}{2} m\omega^2 x^2 = K$$

$$(mK)^{1/2}\left(\frac{K}{m\omega^2}\right)^{1/2} = \hbar/2 \quad \Rightarrow \quad K = \frac{\hbar\omega}{2}$$

$$x = \sqrt{\frac{2K}{m\omega^2}} = \sqrt{\frac{\hbar}{m\omega}}$$

2.4 b)



$V(x) = b|x|$

$\Delta p^2 = mK$

$bx = K \Rightarrow x = \frac{K}{b}$

$$\Delta p \Delta x = \sqrt{mK} \cdot \frac{K}{b} = \hbar/2$$

$$K = \left(\frac{\hbar b}{2\sqrt{m}}\right)^{2/3}$$

▪ problem 3

In[134]:= $f = (q[z])^{-1}\, e^{i\, k \frac{(x^2+y^2)}{2\, q[z]}}$

Out[134]= $$\frac{e^{\frac{i\, k\, (x^2+y^2)}{2\, q[z]}}}{q[z]}$$

In[135]:= $2\, i\, k\, \partial_z f + \partial_{x,x} f + \partial_{y,y} f$ // Simplify

Out[135]= $$\frac{e^{\frac{i\, k\, (x^2+y^2)}{2\, q[z]}}\, k\, \left(k\, \left(x^2+y^2\right) - 2\, i\, q[z]\right)\, (-1 + q'[z])}{q[z]^3}$$

▪ problem 4

In[137]:= $\mathrm{Exp}\left[i\, k\, \frac{(x^2+y^2)}{2\, \left(z - i\, k\, w0^2 / 2\right)}\right]\, \mathrm{Exp}\left[-i\, k\, \frac{(x^2+y^2)}{2\, \left(z + i\, k\, w0^2 / 2\right)}\right]$ // Simplify

Out[137]= $e^{-\frac{2\, k^2\, w0^2\, (x^2+y^2)}{k^2\, w0^4 + 4\, z^2}}$

In[139]:= $\mathrm{Solve}\left[-\frac{2\, k^2\, w0^2\, \left(x^2+y^2\right)}{k^2\, w0^4 + 4\, z^2} == -2\, \left(x^2+y^2\right) / w^2,\ w\right]$

Out[139]= $$\left\{\left\{w \to -\frac{\sqrt{k^2\, w0^4 + 4\, z^2}}{k\, w0}\right\},\ \left\{w \to \frac{\sqrt{k^2\, w0^4 + 4\, z^2}}{k\, w0}\right\}\right\}$$

In[140]:= $\sqrt{\frac{k^2\, w0^4 + 4\, z^2}{(k\, w0)^2}}$ // Simplify

Out[140]= $$\sqrt{w0^2 + \frac{4\, z^2}{k^2\, w0^2}}$$

In[142]:=
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Plot[Sqrt[1 + z^2], {z, 0, 5}, AxesLabel → {"z/z_R", "w/w_0"}]
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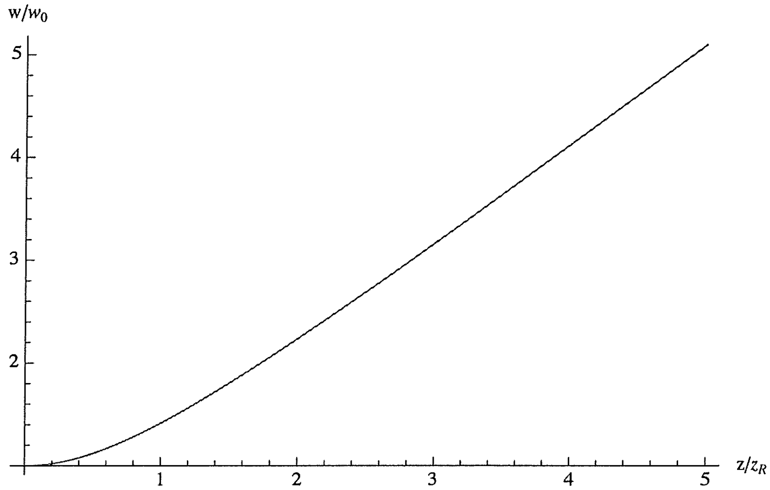
Out[142]=



■ problem 6 (BD 2.2b)

```
parts[u_, v_, n_] := -D[v, {x, n - 1}] D[u, x] (*function to integrate by parts*)
```

part a

In[182]:=
```
parts[x^2 psic[x], psi[x], 2] - parts[x^2 psi[x], psic[x], 2] // Simplify
```

Out[182]= $-2\,x\,\mathrm{psic}[x]\,\mathrm{psi}'[x] + 2\,x\,\mathrm{psi}[x]\,\mathrm{psic}'[x]$

part b

In[178]:=
```
parts[x psic'[x], psi[x], 2] - parts[x psi[x], psic[x], 3] // Simplify
```

Out[178]= $-\mathrm{psi}'[x]\,\mathrm{psic}'[x] + \mathrm{psi}[x]\,\mathrm{psic}''[x]$

In[179]:=
```
-psi'[x] psic'[x] + parts[psi[x], psic[x], 2]
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Out[179]= $-2\,\mathrm{psi}'[x]\,\mathrm{psic}'[x]$

part c

In[183]:=
```
parts[psic'[x], psi[x], 3] - parts[psi'[x], psic[x], 3]
```

Out[183]= 0

■ problem 7 (BD 2.3)

In[185]:= $$\mathrm{Integrate}\left[\left(\frac{\mathrm{Exp}\left[\frac{-p^2}{2\,\sigma^2\,\hbar^2}\right]}{\left(\pi\,\sigma^2\,\hbar^2\right)^{1/4}}\right)^2 p^2,\ \{p, -\infty, \infty\},\ \mathrm{Assumptions} \to \left\{2\,\sigma^2\,\hbar^2 > 0\right\}\right]$$

Out[185]= $$\frac{\sigma^2\,\hbar^2}{2}$$

In[189]:= $$\text{Integrate}\left[\left(\frac{\text{Exp}\left[\frac{-p^2}{2\,\sigma^2\,\hbar^2}\right]}{\left(\pi\,\sigma^2\,\hbar^2\right)^{1/4}}\right)\frac{\text{Exp}[i\,p\,x/\hbar]}{(2\,\pi\,\hbar)^{1/2}},\ \{p,-\infty,\infty\},\ \text{Assumptions}\rightarrow\{\sigma>0,\hbar>0\}\right]$$

Out[189]= $$\frac{e^{-\frac{1}{2}x^2\sigma^2}\sqrt{\sigma}}{\pi^{1/4}}$$

In[190]:= $$\text{Integrate}\left[\%^2\,x^2,\ \{x,-\infty,\infty\},\ \text{Assumptions}\rightarrow\{\sigma>0,\hbar>0\}\right]$$

Out[190]= $$\frac{1}{2\,\sigma^2}$$

part b

In[192]:= $$\text{Integrate}\left[\left(\frac{\text{Exp}\left[\frac{-p^2}{2\,\sigma^2\,\hbar^2}\right]}{\left(\pi\,\sigma^2\,\hbar^2\right)^{1/4}}\right)\frac{\text{Exp}\left[i\,p\,x/\hbar-i\,\frac{p^2}{2\,m\,\hbar}\,t\right]}{(2\,\pi\,\hbar)^{1/2}},\ \{p,-\infty,\infty\},\ \text{Assumptions}\rightarrow\{\sigma>0,\hbar>0,m>0,t>0\}\right]$$

Out[192]= $$\frac{e^{-\frac{m\,x^2\,\sigma^2}{2\,m+2\,i\,t\,\sigma^2\,\hbar}}}{\pi^{1/4}\sqrt{\frac{1}{\sigma}+\frac{i\,t\,\sigma\,\hbar}{m}}}$$

In[193]:= $$\text{Integrate}\left[\frac{e^{-\frac{m\,x^2\,\sigma^2}{2\,m+2\,i\,t\,\sigma^2\,\hbar}}}{\pi^{1/4}\sqrt{\frac{1}{\sigma}+\frac{i\,t\,\sigma\,\hbar}{m}}}\ \frac{e^{-\frac{m\,x^2\,\sigma^2}{2\,m-2\,i\,t\,\sigma^2\,\hbar}}}{\pi^{1/4}\sqrt{\frac{1}{\sigma}-\frac{i\,t\,\sigma\,\hbar}{m}}}\ x^2,\ \{x,-\infty,\infty\},\ \text{Assumptions}\rightarrow\{\sigma>0,\hbar>0,m>0,t>0\}\right]$$

Out[193]= $$\frac{m^2+t^2\,\sigma^4\,\hbar^2}{2\,m^2\,\sigma^2}$$