HW #5 solns

1) BD 4.2

a) $\psi(x,t) = \cos\theta\, \phi_0\, e^{-iE_0 t/\hbar} + \sin\theta\, \phi_1\, e^{-iE_1 t/\hbar}$

b) $\langle E \rangle = \cos^2\theta \frac{\hbar\omega}{2} + \sin^2\theta \frac{3\hbar\omega}{2} = \frac{\hbar\omega}{2}\left(3\sin^2\theta + \cos^2\theta\right)$

$\langle E^2 \rangle = \cos^2\theta \frac{\hbar^2\omega^2}{4} + \sin^2\theta \frac{9\hbar^2\omega^2}{4} = \frac{\hbar^2\omega^2}{4}\left(9\sin^2\theta + \cos^2\theta\right)$

~~$\Delta E^2 = \frac{\hbar^2\omega^2}{4}\left(3\sin^2\theta + \cos^2\theta\right.$~~

$\Delta E^2 = \frac{\hbar^2\omega^2}{4} \cos^2\theta \sin^2\theta$

all time-independent

c) $\langle x \rangle = \int \psi^* x \psi\, dx \overset{M}{=} \sqrt{\frac{\hbar}{m\omega}} \left(\frac{2\sin 2\theta}{3 - \cos 2\theta}\right) \cos\omega t$

$\langle x^2 \rangle = \frac{\hbar}{m\omega}\left(\frac{7 - 5\cos 2\theta}{6 - 2\cos 2\theta}\right)$ time independent

$\Delta x = \langle x^2 \rangle - \langle x \rangle^2$

2) 4.3 a) $E = \hbar\omega\left(n_x + n_y + n_z + 3/2\right)$

let $M = n_x + n_y + n_z$

$\sum_{n_z=0}^{M} \sum_{n_y=0}^{M-n_z} =$ [illegible] $\sum_{n_z=0}^{M} (M - n_z + 1)$

$= \frac{(M+1)(M+2)}{2}$

b) $E = \hbar\omega_x(n_x + 1/2) + \hbar\omega_y(n_y + 1/2) + \hbar\omega_z(n_z + 1/2)$

3) 4.5

$$-\frac{\hbar^2}{2m}\psi'' - \frac{A}{x}\psi = E\psi$$

$$x e^{-x/a}\left(\frac{\hbar^2 + 2a^2 mE}{a}\right) + \frac{e^{-x/a}}{a}\left(2a^2 Am - 2a\hbar^2\right) = 0$$

both terms must be seperately equal

$$\therefore\ a = \frac{\hbar^2}{mA} = \frac{(\hbar c)^2}{mc^2\, ke^2} = \frac{\hbar c}{mc^2 \alpha} = 0.53 Å$$

$$E = \frac{\hbar^2}{2ma^2} = \frac{\hbar^2 (mc^2\alpha)^2}{2m(\hbar c)^2} = \frac{1}{2}\alpha^2 mc^2 = 13.6\,eV$$

c) $\psi = \sqrt{\frac{4}{a^3}}\, x e^{-x/a}$

d) $\langle x \rangle = \frac{3a}{2}$ $\langle \frac{1}{x} \rangle = \frac{1}{a}$

$$\langle K \rangle = E - \langle V \rangle = -\frac{1}{2}\alpha^2 mc^2 + \frac{ke^2}{a} = +\frac{1}{2}\alpha mc^2$$

$$\langle K \rangle = -\frac{1}{2}\langle V \rangle \quad \text{Virial Thm}$$

4) 4.6 $-\frac{\hbar^2}{2m}\psi'' + \alpha\,\delta(x)\psi = E\psi$

$$-\frac{\hbar^2}{2m}\left(\psi_>'(0) - \psi_<'(0)\right) = -\alpha\,\psi(0)$$

$$\psi = e^{-Kx}$$

$$-\frac{\hbar^2}{2m}(-K - K) = -\alpha \Rightarrow K = \frac{-m\alpha}{\hbar^2} = \sqrt{\frac{-2mE}{\hbar^2}}$$

$$E = -\frac{\hbar^2 K^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{m^2\alpha^2}{\hbar^4}\right) = \frac{-m\alpha^2}{2\hbar^2}$$

one bound state

4b) $\psi = \begin{cases} e^{-K|x|}(\pm) & |x| > d/2 \\ A\cosh Kx + B\sinh Kx & |x| < d/2 \end{cases}$ $\quad$ + if symmetric, − if antisymmetric

Quantization cond

sym:

$$-\frac{\hbar^2}{2m}\left(-K e^{-Kd/2} - \frac{K e^{-Kd/2}}{\cosh Kd/2}\sinh\frac{Kd}{2}\right) = -\alpha\, e^{-Kd/2}$$

$$K\left(1+\tanh\frac{Kd}{2}\right) = -\frac{2m\alpha}{\hbar^2}$$

antisym: $\frac{Kd}{2}\left(1+\coth\frac{Kd}{2}\right) = -\frac{m\alpha d}{\hbar^2}$

only soln if $m\alpha d/\hbar^2 > 1$

$\therefore$ only 1 soln for $d < \frac{\hbar^2}{m\alpha}$

2 soln otherwise

5) $\displaystyle\int_{x_0}^{x_1} k\,dx = n\pi$, x_0 & x_1 are classical turning pts

$$k = \sqrt{\frac{2m(E-V)}{\hbar^2}} = \sqrt{\frac{2mE}{\hbar^2}\left(1 - \frac{b|x|}{E}\right)}$$

$$\left(\frac{2mE}{\hbar^2}\right)^{1/2} \frac{E}{b} \int_{-1}^{1} du \sqrt{1-|u|} = \left(\frac{2mE}{\hbar^2}\right)^{1/2} \frac{E}{b} \cdot \frac{4}{3} = n\pi$$

$$\therefore\ E = \left(\frac{9\pi^2\hbar^2c^2b^2}{32mc^2}\right)^{1/3} n^{2/3}$$

$$= 314\ \text{nK}\ n^{2/3}$$

6) see Mathematica

- **Prob 1**

In[1]:= $\frac{(9\,\mathrm{Sin}[th]^2+\mathrm{Cos}[th]^2)}{4}-\frac{(3\,\mathrm{Sin}[th]^2+\mathrm{Cos}[th]^2)^2}{4}$ // Simplify

Out[1]= $\mathrm{Cos}[th]^2\,\mathrm{Sin}[th]^2$

In[2]:= $Assumptions = {ℏ > 0, m > 0, ω > 0, l > 0}

Out[2]= {ℏ > 0, m > 0, ω > 0, l > 0}

define the simple harmonic oscillator wavefunctions

In[3]:= sho[n_] := With$\left[\left\{y=\frac{x}{\sqrt{\frac{\hbar}{m\,\omega}}}\right\},\ \mathrm{HermiteH}[n,y]\ \mathrm{Exp}[-y^2/2]\right]$

In[4]:= conj[x_] := x /. Complex[a_, b_] → Complex[a, -b]

In[5]:= psi = Cos[θ] sho[0] + Sin[θ] sho[1] $e^{-i\,\omega\,t}$

Out[5]= $e^{-\frac{m x^2 \omega}{2\hbar}}\,\mathrm{Cos}[\theta]+\frac{2\,e^{-i t \omega-\frac{m x^2 \omega}{2\hbar}}\,x\,\mathrm{Sin}[\theta]}{\sqrt{\frac{\hbar}{m\,\omega}}}$

In[6]:= $\frac{\mathrm{Integrate}[\mathrm{conj}[psi]\,x\,psi,\{x,-\infty,\infty\}]}{\mathrm{Integrate}[\mathrm{conj}[psi]\,psi,\{x,-\infty,\infty\}]}$

Out[6]= $-\frac{2\sqrt{m\,\omega\,\hbar}\,\mathrm{Cos}[t\,\omega]\,\mathrm{Sin}[2\,\theta]}{m\,\omega\,(-3+\mathrm{Cos}[2\,\theta])}$

In[7]:= % // PowerExpand

Out[7]= $-\frac{2\sqrt{\hbar}\,\mathrm{Cos}[t\,\omega]\,\mathrm{Sin}[2\,\theta]}{\sqrt{m}\,\sqrt{\omega}\,(-3+\mathrm{Cos}[2\,\theta])}$

In[8]:= $\frac{\mathrm{Integrate}[\mathrm{conj}[psi]\,x^2\,psi,\{x,-\infty,\infty\}]}{\mathrm{Integrate}[\mathrm{conj}[psi]\,psi,\{x,-\infty,\infty\}]}$ // PowerExpand

Out[8]= $-\frac{\hbar\,(7-5\,\mathrm{Cos}[2\,\theta])}{2\,m\,\omega\,(-3+\mathrm{Cos}[2\,\theta])}$

In[9]:= $-\frac{\hbar\,(7-5\,\mathrm{Cos}[2\,\theta])}{2\,m\,\omega\,(-3+\mathrm{Cos}[2\,\theta])}-\left(-\frac{2\sqrt{\hbar}\,\mathrm{Cos}[t\,\omega]\,\mathrm{Sin}[2\,\theta]}{\sqrt{m}\,\sqrt{\omega}\,(-3+\mathrm{Cos}[2\,\theta])}\right)^2$ // Simplify

Out[9]= $\frac{\hbar\,(21-22\,\mathrm{Cos}[2\,\theta]+5\,\mathrm{Cos}[2\,\theta]^2-8\,\mathrm{Cos}[t\,\omega]^2\,\mathrm{Sin}[2\,\theta]^2)}{2\,m\,\omega\,(-3+\mathrm{Cos}[2\,\theta])^2}$

- **prob 2**

In[10]:= `Sum[(m - nz + 1), {nz, 0, m}]`

Out[10]= $\frac{1}{2}(1+m)(2+m)$

- **prob 3**

In[21]:= `psi[x] = x Exp[-x / a]`

Out[21]= $e^{-\frac{x}{a}} x$

In[23]:= $\frac{-\hbar^2}{2\,m}$ `D[psi[x], {x, 2}]` $- \frac{AA}{x}$ `psi[x] == EE psi[x] // Simplify`

Out[23]= $\frac{e^{-\frac{x}{a}}\left(2\,a^2\,m\,(AA + EE\,x) - 2\,a\,\hbar^2 + x\,\hbar^2\right)}{a} == 0$

In[24]:= `Collect[%, x]`

Out[24]= $\frac{e^{-\frac{x}{a}}\,x\left(2\,a^2\,EE\,m + \hbar^2\right)}{a} + \frac{e^{-\frac{x}{a}}\left(2\,a^2\,AA\,m - 2\,a\,\hbar^2\right)}{a} == 0$

In[25]:= `Integrate[psi[x]^2, {x, 0, ∞}]`

Out[25]= ConditionalExpression$\left[\frac{a^3}{4}, \text{Re}[a] > 0\right]$

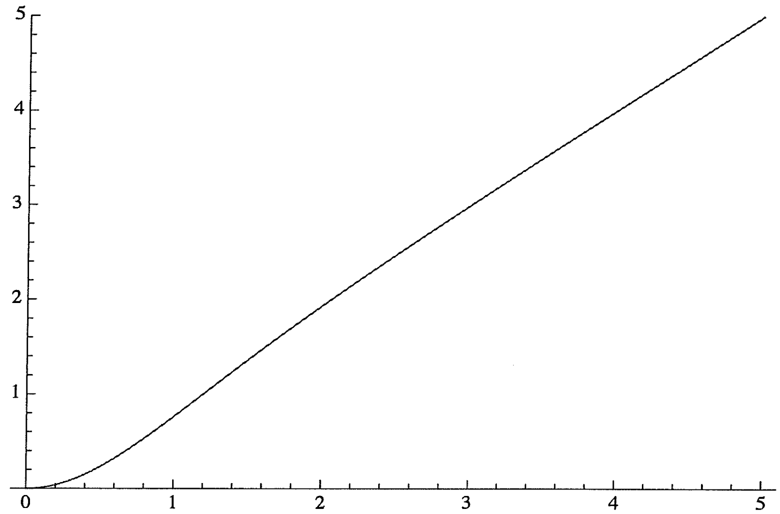
In[27]:= Integrate$\left[\left(\sqrt{\frac{4}{a^3}}\ \text{psi}[x]\right)^2 x^{-1}, \{x, 0, \infty\}\right]$

Out[27]= ConditionalExpression$\left[\frac{1}{a}, \text{Re}[a] > 0\right]$

▪ Prob 4

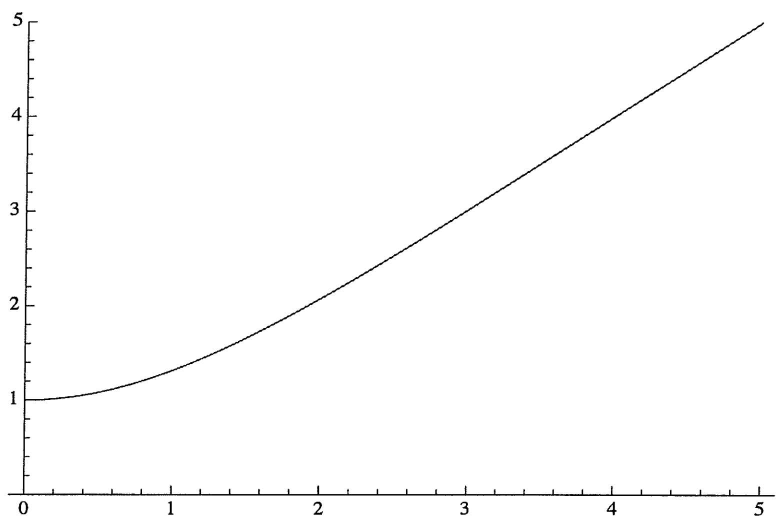
In[30]:= `Plot[x Tanh[x], {x, 0, 5}, PlotRange → {0, 5}]`

Out[30]=



In[29]:= `Plot[x Coth[x], {x, 0, 5}, PlotRange → {0, 5}]`

Out[29]=



▪ Prob 5

In[127]:=

`Integrate[`$\sqrt{1 - \mathrm{Abs}[x]}$`, {x, -1, 1}]`

Out[127]= $\frac{4}{3}$

In[32]:= $\mathbf{mcsq} = 85 \times 931.5 \times 10^6\ \mathbf{eV}\ 11\,600\ \frac{\mathbf{kelvin}}{\mathbf{eV}};$

$\mathbf{hbarc} = 1973\ \mathbf{eV}\ 11\,600\ \mathbf{kelvin} / \mathbf{eV}\ \mathbf{Angstrom};\ \mathbf{b} = \frac{14 \times 10^{-3}\ \mathbf{kelvin} / \mathbf{cm}}{10^8\ \mathbf{Angstrom} / \mathbf{cm}};$

In[34]:= $\frac{b^{2/3}\,\text{hbarc}^{2/3}\,\left(\frac{3\pi}{2}\right)^{2/3}}{2\,\text{mcsq}^{1/3}}$ // PowerExpand

Out[34]= 3.14234×10^{-7} kelvin

- **Prob 6: Determine lowest 5 energy levels**

In[725]:= a = $\left(\frac{\text{hbarc}^2}{\text{mcsq b}}\right)^{1/3}$ // N // PowerExpand

Out[725]= 1597.08 Angstrom

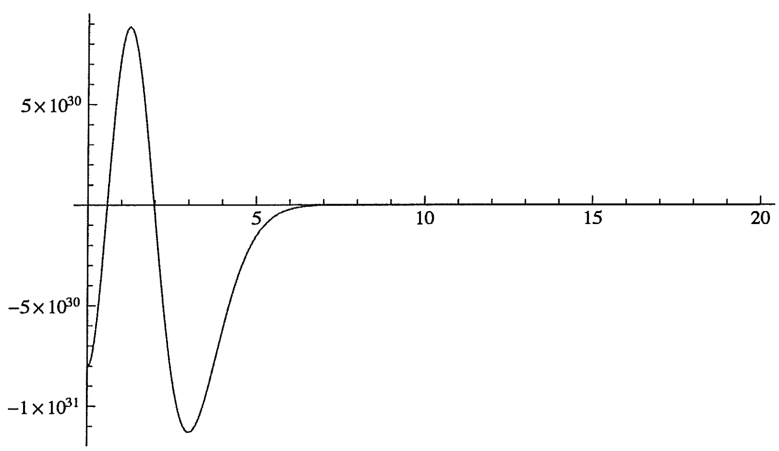
In[727]:= $\frac{\text{hbarc}^2}{\text{mcsq}\,a^2}$

Out[727]= 2.23592×10^{-7} kelvin

In[122]:=
```
With[{ϵ = 3.8257, y0 = 22},
  sol = NDSolve[{-1/2 psi''[y] + Abs[y] psi[y] == ϵ psi[y], psi[y0] == 1,
     psi'[y0] == Sqrt[2 y0] psi[y0]}, psi, {y, y0, 0}];
  Print[psi'[0]/psi[0] /. sol[[1]] // InputForm];
  Plot[psi[Abs[y]] /. sol, {y, 0, 20}, PlotRange → All]
 ]
```

-0.00011690177549852778

Out[122]=



By guessing various values, get the following lowest 5 energy levels

In[125]:= **{.808617, 1.8557, 2.5781, 3.2446, 3.8257} 223.5 nK**

Out[125]= {180.726 nK, 414.749 nK, 576.205 nK, 725.168 nK, 855.044 nK}

from Problem 5

In[35]:=
```
314. nK Range[1, 5]^(2/3)
```
Out[35]=
```
{314. nK, 498.444 nK, 653.146 nK, 791.23 nK, 918.142 nK}
```

ratio of quantum/classical

In[36]:=
```
{180.7258995` nK, 414.74895` nK, 576.2053500000001` nK, 725.1681` nK, 855.04395` nK} /
  {314.` nK, 498.4439303180147` nK, 653.1463204382978` nK,
   791.2304193339803` nK, 918.1415697988398` nK}
```
Out[36]=
```
{0.57556, 0.832087, 0.882199, 0.916507, 0.931277}
```

below is linear interpolation function I used to use my last two guesses to predict the next

In[79]:= guess[ϵ1_, q1_, ϵ2_, q2_] := $\frac{\epsilon 2\ q1 - \epsilon 1\ q2}{q1 - q2}$

In[123]:=
```
guess[3.823, -.02076, 3.8257, -.000117]
```
Out[123]=
```
3.82572
```