HW 6 Sol

5.1) a) $e^{-ix_0 p/\hbar}\psi = e^{-x_0 \partial_x}\psi$

$$= \sum_n \frac{(-x_0)^n \psi^{(n)}(x)}{n!}$$ nth derivative

compare Taylor Series $\psi(x - x_0) = \sum_n \frac{(-x_0)^n}{n!}\psi^{(n)}(x)$

$$= \psi(x - x_0)$$

Alternate

$$e^{-ix_0 p/\hbar}|\psi\rangle = \int \frac{dp}{(2\pi\hbar)^{1/2}} e^{-i\frac{x_0 p}{\hbar}} e^{i\frac{px}{\hbar}} \varphi(p)$$

$$= \int \frac{dp}{(2\pi\hbar)^{1/2}} e^{ip(x-x_0)/\hbar} \varphi(p)$$

$$= \psi(x - x_0)$$

b) $$e^{-i\phi L_z/\hbar}\psi(r,\theta) = e^{-\phi\frac{\partial}{\partial\theta}}\psi(r,\theta)$$

$$= \sum_n \frac{(-\phi)^n}{n!}\partial_\theta^n \psi(r,\theta) = \psi(r, \theta - \phi)$$

or, $\psi(r,\theta) = \sum_m \psi_m(r) e^{im\theta}$

$$e^{-\phi\partial_\theta} e^{im\theta} = \sum_n \frac{(-\phi)^n}{n!}(im)^n e^{im\theta} = e^{-im\phi} e^{im\theta}$$

$$= e^{im(\theta-\phi)}$$

$$\therefore e^{-i\phi L_z/\hbar}\psi(r,\theta) = \sum_m \psi_m(r) e^{im(\theta-\phi)} = \psi(r, \theta - \phi)$$

5.2) $$e^{-iH(t-t_0)/\hbar}\psi(t_0) = \sum_n \left(\frac{-iH(t-t_0)}{\hbar}\right)^n \frac{\psi(t_0)}{n!}$$

$$H\psi = i\hbar\partial_t\psi$$

$$= \sum_n (t-t_0)^n \frac{\partial_t^n \psi(t_0)}{n!} = \psi(t)$$

or, $$\psi(t) = \sum_n e^{-iE_n(t-t_0)/\hbar} |n\rangle\langle n|\psi(t_0)\rangle$$

(where $H|n\rangle = E_n|n\rangle$)

$$= \sum_n e^{-iH(t-t_0)/\hbar} |n\rangle\langle n|\psi(t_0)\rangle$$

$$= e^{-iH(t-t_0)/\hbar}\psi(t_0)$$

or, $$\cancel{e^{-iH}}\ \psi(t_0) = \int \frac{d\omega}{\sqrt{2\pi}} \tilde{\psi}(\omega) e^{-i\omega t_0}$$

$$e^{-iH(t-t_0)/\hbar} e^{-i\omega t_0} = e^{(t-t_0)\partial_t} e^{-i\omega t_0}$$

$$= \sum_n \frac{(t-t_0)^n}{n!}(-i\omega)^n e^{-i\omega t_0}$$

$$= e^{-i\omega t}$$

$$\therefore\ e^{-iH(t-t_0)/\hbar}\psi(t_0) = \int \frac{d\omega}{2\pi}\tilde{\psi}(\omega) e^{-i\omega t}$$

$$= \psi(t)$$

b) $$U^\dagger \cdot U = e^{iH(t-t_0)/\hbar} e^{-iH(t-t_0)/\hbar} = 1$$

$$\therefore\ U^\dagger = U^{-1}$$

5.3)

$$a(t) = \langle \psi(t) | A | \psi(t) \rangle$$

$$= \langle \psi(0) | U^{+}(t) A U(t) | \psi(0) \rangle$$

$$= \langle \psi(0) | A(t) | \psi(0) \rangle$$

$$i\hbar \frac{dA}{dt} = i\hbar \underbrace{\left(\frac{d}{dt} U^{+}\right)}_{\frac{iHU^{+}}{\hbar}} A U + i\hbar U^{+} A \underbrace{\frac{dU}{dt}}_{\frac{-iHU}{\hbar}}$$

$$= -H A(t) + A(t) H$$

$$= [A, H]$$

5.4) $\langle \psi_2 | H | \psi_1 \rangle = E_2 \langle \psi_2 | \psi_1 \rangle = E_1 \langle \psi_2 | \psi_1 \rangle$

$$\therefore (E_2 - E_1) \langle \psi_2 | \psi_1 \rangle = 0 \Rightarrow \langle \psi_2 | \psi_1 \rangle = 0$$

b) $\left\langle \frac{\psi_1 - \psi_2}{\sqrt{2}} \right| H \left| \frac{\psi_1 - \psi_2}{\sqrt{2}} \right\rangle = \frac{E_1}{2} + \frac{E_2}{2} = \frac{E_1 + E_2}{2}$

$$\langle \quad | H^2 | \quad \rangle = \frac{E_1^2}{2} + \frac{E_2^2}{2}$$

$$\Delta E = \left(\frac{E_1^2}{2} + \frac{E_2^2}{2} - \frac{(E_1 + E_2)^2}{4} \right)^{1/2} = \left(\frac{(\hbar\omega)^2}{4} \right)^{1/2} = \frac{\hbar\omega}{2}$$

c) $|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{-iE_1 t/\hbar} |\psi_1\rangle - e^{-iE_2 t/\hbar} |\psi_2\rangle \right)$

d) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 \Rightarrow \lambda = \pm 1$

e) Eigenvectors $|+1\rangle$ $\quad (-1)\langle\psi_1|+1\rangle + \langle\psi_2|+1\rangle = 0$

$$\therefore\ |+1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$|-1\rangle$ $\quad \langle\psi_1|-1\rangle + \langle\psi_2|-1\rangle = 0$

$$|-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

f) $P_{-1} = |\langle -1|\psi\rangle|^2$

$$\langle -1|\psi\rangle = \frac{1}{\sqrt{2}} e^{-iE_1t/\hbar} \underbrace{\langle -1|\psi_1\rangle}_{1/\sqrt{2}} - \frac{1}{\sqrt{2}} e^{-iE_2t/\hbar} \underbrace{\langle -1|\psi_2\rangle}_{-1/\sqrt{2}}$$

$$= e^{-i\frac{(E_1+E_2)}{2\hbar}}\left(\frac{1}{2}e^{-i\omega t} + \frac{1}{2}e^{i\omega t}\right)$$

$$= e^{-i\ldots}\cos\omega t$$

$$\therefore\ P_{-1} = \cos^2\omega t$$

5) $H = \mathrm{diag}(0, \Delta, 0) + V$

$$= \begin{pmatrix} 0 & \epsilon & 0 \\ \epsilon^* & \Delta & \epsilon \\ 0 & \epsilon^* & 0 \end{pmatrix}$$

Mathematica let $\tan 2\theta = \frac{\sqrt{8}|\epsilon|}{\Delta}$, $\Omega = \sqrt{8|\epsilon|^2 + \Delta^2}$

$E = 0$, $|0\rangle = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$ $\quad E = -\Omega\sin^2\theta$, $|-\rangle = \begin{pmatrix} \cos\theta/\sqrt{2} \\ -e^{-i\phi}\sin\theta \\ \cos\theta/\sqrt{2} \end{pmatrix}$

$E = \Omega\cos^2\theta$, $|+\rangle = \begin{pmatrix} \sin\theta/\sqrt{2} \\ e^{-i\phi}\cos\theta \\ \sin\theta/\sqrt{2} \end{pmatrix}$

dark state is $|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$V|0\rangle = \begin{pmatrix} 0 & \epsilon & 0 \\ \epsilon^* & 0 & \epsilon \\ 0 & \epsilon^* & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0$$

in that state, there is no interaction with the light!

In[235]:= $\{eval, kets\} = \text{Eigensystem}\left[\begin{pmatrix} 0 & e\,e^{i\phi} & 0 \\ e\,e^{-i\phi} & \Delta & e\,e^{-i\phi} \\ 0 & e\,e^{i\phi} & 0 \end{pmatrix} /. \left\{e \to \frac{\text{Sin}[2\theta]}{\sqrt{8}}\,\Omega,\ \Delta \text{->} \text{Cos}[2\theta]\,\Omega\right\}\right]$ //
TrigFactor // Simplify

Out[235]= $\left\{\left\{0, -\Omega\,\text{Sin}[\theta]^2, \Omega\,\text{Cos}[\theta]^2\right\}, \left\{\{-1, 0, 1\}, \left\{1, -\sqrt{2}\,e^{-i\phi}\,\text{Tan}[\theta], 1\right\}, \left\{1, \sqrt{2}\,e^{-i\phi}\,\text{Cot}[\theta], 1\right\}\right\}\right\}$

note: *Mathematica* gives the eigenvectors as row vectors. So the individual kets are kets[[1]], kets[[2]] etc. The bras are conj[kets[[1]]] and so on. When *Mathematica* calculates the eigenvectors and eigenvalues of a numerical matrix, it gives properly normalized eigenvectors. However, for an analytical matrix they are not normalized so you have to take care of the normalization yourself.

In[236]:=
```
norm = Table[conj[kets[[i]]].kets[[i]], {i, 3}] //
  Simplify (*Calculate the norms of the kets*)
```

Out[236]= $\left\{2, 2\,\text{Sec}[\theta]^2, 2\,\text{Csc}[\theta]^2\right\}$

In[237]:= $\text{kets} = \frac{\text{kets}}{\sqrt{\text{norm}}}$ // PowerExpand // Simplify

Out[237]= $\left\{\left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{\text{Cos}[\theta]}{\sqrt{2}}, -e^{-i\phi}\,\text{Sin}[\theta], \frac{\text{Cos}[\theta]}{\sqrt{2}}\right\}, \left\{\frac{\text{Sin}[\theta]}{\sqrt{2}}, e^{-i\phi}\,\text{Cos}[\theta], \frac{\text{Sin}[\theta]}{\sqrt{2}}\right\}\right\}$