# HW #8 Solns

1) BP 7.1

$$[A,BC] = ABC - BCA \overbrace{- BAC + BAC}^{=0}$$

$$= [A,B]C + B[A,C]$$

$$d_n = [A,B^n] = [A,B^{n-1}]B + B^{n-1}[A,B]$$

$$= d_{n-1}B + B^{n-1}[A,B]$$

Check $d_n = \sum_{s=0}^{n-1} B^s[A,B]B^{n-1-s}$

$$d_n = \sum_{s=0}^{n-2} B^s[A,B]B^{n-2-s}\cdot B + B^{n-1}[A,B]$$

$$= d_{n-1}B + B^{n-1}[A,B] \quad ✓$$

∴ true by induction

$\cancel{[A,BC]}+$

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$$

$$= [A, BC - CB] + \ldots$$

$$= [A,B]C + B[A,C] - [A,C]B - C[A,B]$$

$$+ [B,C]A + C[B,A] - [B,A]C - A[B,C]$$

$$+ [C,A]B + A[C,B] - [C,B]A - B[C,A]$$

$$= 2B[A,C] - [A,C]2B + 2[B,C]A -$$

$$([A,B],C] + [[B,C],A] + [[C,A],B] =$$

$$\begin{aligned}
&ABC - BAC - CAB + CBA \\
&+ BCA - CBA - ABC + ACB \quad = 0 \\
&+ CAB - ACB - BCA + BAC
\end{aligned}$$

2) BD 7.2

$$F = e^{tA} e^{tB} = \sum_{mn} \frac{(tA)^m}{m!} \frac{(tB)^n}{n!} =$$

$$\cancel{\frac{dB}{dt} = AF + FB}$$

$$\frac{dF}{dt} = \sum_{mn} (m+n) t^{m+n-1} \frac{A^m B^n}{m!\,n!}$$

$$= AF + FB = AF + e^{tA} B e^{tB}$$

$(A,B) = -[B,A]$

$$e^{tA} B = \sum_m \frac{t^m A^m}{m!} B = \sum_m \frac{t^m}{m!} \left( BA^m - [B, A^m] \right)$$

$[A,[A,B]] = 0$

$[B,[A,B]] = 0$

$$= Be^{tA} - \sum_m \frac{t^m}{m!} \underbrace{\sum_{s=0}^{m-1} A^s [B,A] A^{m-s-1}}_{= [B,A] A^{m-1}}$$

because $[A,[B,A]] = 0$

$$= Be^{tA} - \sum_m \frac{t^m}{m!} m [B,A] A^{m-1}$$

$$= Be^{tA} - t[B,A] e^{tA}$$

$$\therefore \frac{dF}{dt} = (A + B + t[A,B]) F$$

Int from 0 to 1

$$F = e^{(A+B)t} e^{t^2/2 [A,B]} = e^{tA} e^{tB}$$

$$t = 1, \quad e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$$

3) BD 7.6

$$[p_x, f]\psi = p_x f\psi - f p_x \psi$$

$$p_x = \frac{x}{r} p_r \qquad \left(\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} = \frac{x}{r}\frac{\partial}{\partial r}\right)$$

$$= \frac{x}{r}\Big((p_r f)\psi + f p_r \psi\Big) - f\frac{x}{r} p_r \psi$$

$$= -i\hbar \frac{x}{r} f'$$

b) $A_x = p_x - i\lambda x f$

$$\langle\psi|A_x^\dagger A_x|\psi\rangle = \langle\psi|(p_x + i\lambda x f)(p_x - i\lambda x f)|\psi\rangle$$

$$= \langle p_x^2\rangle + i\lambda\langle[xf, p_x]\rangle + \lambda^2\langle x^2 f^2\rangle$$

$$[p_x, xf] = [p_x, x]f + x[p_x, f]$$

$$= -i\hbar f + x\left(-i\hbar\frac{x}{r}f'\right) = -i\hbar\left(f + \frac{x^2}{r}f'\right)$$

$$= \langle p_x^2\rangle + \lambda\hbar\left\langle f + \frac{x^2}{r}f'\right\rangle + \lambda^2\langle x^2 f^2\rangle$$

add $y, z$, to get

$$\sum_i \langle\psi|A_i^\dagger A_i|\psi\rangle = \langle p^2\rangle + \lambda\hbar\langle 3f + rf'\rangle + \lambda^2\langle r^2 f^2\rangle > 0$$

$$\therefore 4\langle p^2\rangle\langle r^2 f^2\rangle > \hbar^2\langle 3f + rf'\rangle^2$$

c) $f = 1 \qquad \langle p^2\rangle\langle r^2\rangle \geq \frac{9}{4}\hbar^2$

$f = 1/r \qquad \langle p^2\rangle \geq \frac{\hbar^2}{4}\left\langle\frac{2}{r}\right\rangle^2 = \hbar^2\left\langle\frac{1}{r}\right\rangle^2$

3c cont) $f = 1/r^2 \quad \langle p^2\rangle\langle \frac{1}{r^2}\rangle \geq \frac{\hbar^2}{4}\left\langle \frac{3}{r^2} - \frac{2}{r^2}\right\rangle^2$

$$\langle p^2\rangle \geq \frac{\hbar^2}{4}\left\langle \frac{1}{r^2}\right\rangle$$

d) $\langle E\rangle = \frac{1}{2m}\langle p^2\rangle + \frac{1}{2}m\omega^2\langle r^2\rangle > \frac{9\hbar^2}{8m\langle r^2\rangle} + \frac{m\omega^2\langle r^2\rangle}{2}$

minimized for $\langle r^2\rangle = \left(\frac{9}{4}\frac{\hbar^2}{m^2\omega^2}\right)^{1/2} = \frac{3}{2}\frac{\hbar}{m\omega}$

$\cancel{\langle p^2\rangle = \frac{3}{2}\hbar\omega \cdot 2m - \frac{1}{2}m\omega^2\left(\frac{3}{2}\frac{\hbar}{m\omega}\right)\cdot 2m}$

$\frac{1}{2}m\omega^2\langle r^2\rangle = \frac{3}{4}\hbar\omega \Rightarrow \frac{\langle p^2\rangle}{2m} = 3/4\,\hbar\omega$

$\therefore \quad \langle p^2\rangle = \frac{9}{4}\frac{\hbar^2}{3/2\,\hbar/m\omega} = \frac{3}{2}m\hbar\omega$ is min poss value.

$\therefore \quad A_x|\psi\rangle = 0$

$\therefore \quad (p_x + i\lambda x)|\psi\rangle = 0$ similar for $y, z$

$\therefore \quad \left(-\hbar\frac{d}{dx} + \lambda x\right)\psi = 0 \Rightarrow \psi = e^{+\frac{\lambda x^2}{2\hbar}}$

$\therefore \quad \psi = e^{+\frac{\lambda r^2}{2\hbar}}$

$\langle p^2\rangle + 3\lambda\hbar + \lambda^2\langle r^2\rangle = 0$

$\therefore \quad \lambda = \frac{-3\hbar + \sqrt{9\hbar^4 - 4\langle p^2\rangle\langle r^2\rangle}}{2\langle p^2\rangle} = \frac{-3\hbar}{2\langle p^2\rangle}$

$= \frac{-3\hbar}{2\frac{3}{2}m\hbar\omega} = -\frac{1}{m\omega}$

$\therefore \quad \psi = e^{\frac{-\hbar r^3}{2m\omega}}$

c) $\langle E\rangle = \left\langle \frac{p^2}{2m}\right\rangle - e^2\left\langle\frac{1}{r}\right\rangle$

$$\geq \frac{\hbar^2}{2m}\left\langle\frac{1}{r}\right\rangle^2 - e^2\left\langle\frac{1}{r}\right\rangle$$

minimum at $\left\langle\frac{1}{r}\right\rangle = \left(\frac{\cancel{2}me^2}{\hbar^2}\right)^{[illegible]}$

$$\langle p^2\rangle = 2m\left(E + e^2 \cdot \frac{e^2}{2m\hbar^2}\right)$$

$$= 2m\left(-\frac{1}{2}\frac{e^4 m}{\hbar^2} + \frac{me^4}{\hbar^2}\right) = m^2e^4/\hbar^2$$

$$= \hbar^2\left\langle\frac{1}{r}\right\rangle^2$$

$\therefore$ minimum possible value

$\therefore$ $\left(p_x - i\lambda\frac{x}{r}\right)\psi = 0$

$$\left(\frac{x}{r}p_r - i\lambda\frac{x}{r}\right)\psi = 0$$

$$\hbar\frac{d\psi}{dr} + \lambda\psi = 0 \quad \psi = e^{-\lambda r}$$

$$\langle p^2\rangle + \lambda\hbar\left\langle 2/r\right\rangle + \lambda^2 = 0$$

$$\lambda = \frac{-\hbar\langle 2/r\rangle + \sqrt{0}}{2\langle p^2\rangle} = \frac{\hbar}{\hbar^2\langle\frac{1}{r}\rangle} =$$

$\therefore$ $\psi = e^{-r}$

## ■ #4

In[109]:= `{eval, evec} = Eigensystem[h = {{0, -a, -a}, {-a, 0, -a}, {-a, -a, 0}}]`

Out[109]= `{{-2 a, a, a}, {{1, 1, 1}, {-1, 0, 1}, {-1, 1, 0}}}`

In[110]:= $\rho$ `= {{0, 1, 0}, {0, 0, 1}, {1, 0, 0}};`
$\rho$ `- conj[`$\rho^T$`] (*test for Hermitian*)`

Out[111]= `{{0, 1, -1}, {-1, 0, 1}, {1, -1, 0}}`

so $\rho$ is not Hermitian

In[112]:= `h.ρ - ρ.h // Simplify`

Out[112]= `{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}`

so $\rho$ commutes with h

In[113]:= `{ρs, ρkets} = Eigensystem[ρ] // Simplify`

Out[113]= $\left\{\left\{1, \frac{1}{2} i \left(i+\sqrt{3}\right), -\frac{1}{2} i \left(-i+\sqrt{3}\right)\right\}, \left\{\{1, 1, 1\}, \left\{\frac{1}{2} i \left(i+\sqrt{3}\right), -\frac{1}{2} i \left(-i+\sqrt{3}\right), 1\right\}, \left\{-\frac{1}{2} i \left(-i+\sqrt{3}\right), \frac{1}{2} i \left(i+\sqrt{3}\right), 1\right\}\right\}\right\}$

In[114]:= `ρbras = Inverse[ρkets`$^T$`] // Simplify`

Out[114]= $\left\{\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}, \left\{-\frac{1}{6} i \left(-i+\sqrt{3}\right), \frac{1}{6} i \left(i+\sqrt{3}\right), \frac{1}{3}\right\}, \left\{\frac{1}{6} i \left(i+\sqrt{3}\right), -\frac{1}{6} i \left(-i+\sqrt{3}\right), \frac{1}{3}\right\}\right\}$

Check that the eigenstates of $\rho$ are eigenstates of h

In[115]:= `ρbras.h.ρkets`$^T$` // Simplify // MatrixForm`

Out[115]//MatrixForm=

$$\begin{pmatrix} -2a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

so the eigenvectors of $\rho$ indeed are eigenvectors of h with the correct eigenvalues

## ■ #5

In[116]:= `B = i b (ρ - ρ`$^T$`)`

Out[116]= `{{0, i b, -i b}, {-i b, 0, i b}, {i b, -i b, 0}}`

In[117]:= `B - conj[B`$^T$`] // Simplify`

Out[117]= `{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}`

so B is Hermitian. Therefore it is a possible observable.

In[118]:= `B.h - h.B // Simplify`

Out[118]= `{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}` so B commutes with H.

In[119]:= 
```
{Bs, Bkets} = Eigensystem[B] // Simplify
```

Out[119]= $\left\{\left\{0, -\sqrt{3}\ b, \sqrt{3}\ b\right\}, \left\{\{1, 1, 1\}, \left\{\frac{1}{2}\ i\left(i+\sqrt{3}\right), -\frac{1}{2}\ i\left(-i+\sqrt{3}\right), 1\right\}, \left\{-\frac{1}{2}\ i\left(-i+\sqrt{3}\right), \frac{1}{2}\ i\left(i+\sqrt{3}\right), 1\right\}\right\}\right\}$

In[120]:= 
```
Bbras = Inverse[Bkets^T] // Simplify
```

Out[120]= $\left\{\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}, \left\{-\frac{1}{6}\ i\left(-i+\sqrt{3}\right), \frac{1}{6}\ i\left(i+\sqrt{3}\right), \frac{1}{3}\right\}, \left\{\frac{1}{6}\ i\left(i+\sqrt{3}\right), -\frac{1}{6}\ i\left(-i+\sqrt{3}\right), \frac{1}{3}\right\}\right\}$

In[121]:= 
```
Bbras.h.Bkets^T // Simplify
```

Out[121]= $\{\{-2\ a, 0, 0\}, \{0, a, 0\}, \{0, 0, a\}\}$

so the eigenvectors of B are also eigenvectors of H. The energies of H' are just the sum of those of H and B:

In[122]:= 
```
Diagonal[Bbras.(h + B).Bkets^T // Simplify]
```

Out[122]= $\left\{-2\ a, a-\sqrt{3}\ b, a+\sqrt{3}\ b\right\}$