■ Ammonia molecule, $H = H_L + H_R + V$, V treated as perturbation

symmetry arguments ($\langle L_2 \mid V \mid R_1 \rangle = \langle L_1 \mid V \mid R_2 \rangle = f$, $\langle L_2 \mid V \mid L_1 \rangle = \langle R_1 \mid V \mid R_2 \rangle = c$) require the Hamiltonian to have the following structure in the basis $\{ \mid R_1\rangle, \mid L_1\rangle, \mid R_2\rangle, \mid L_2\rangle\}$

$$\mathbf{H} = \begin{matrix} E1 & -A1 & c & f \\ -A1 & E1 & f & c \\ c & f & E2 & -A2 \\ f & c & -A2 & E2 \end{matrix};$$

First separate this out into unperturbed and perturbed parts, keeping the degenerate subspaces as part of H0

$$\mathbf{H0} = \begin{matrix} E1 & -A1 & 0 & 0 \\ -A1 & E1 & 0 & 0 \\ 0 & 0 & E2 & -A2 \\ 0 & 0 & -A2 & E2 \end{matrix};$$

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(V = H - H0) // MatrixForm
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$$\begin{pmatrix} 0 & 0 & c & f \\ 0 & 0 & f & c \\ c & f & 0 & 0 \\ f & c & 0 & 0 \end{pmatrix}$$

Find the eigenvalues and vectors for H0 (note much simpler than full H):

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{e0s, kets} = Eigensystem[H0] // Simplify
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{{-A1 + E1, A1 + E1, -A2 + E2, A2 + E2},
 {{1, 1, 0, 0}, {-1, 1, 0, 0}, {0, 0, 1, 1}, {0, 0, -1, 1}}}
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Note that the eigenstates are even and odd combinations of the Rs and Ls. The first and 3rd states are even, the 2nd and 4th odd.

$$\mathbf{bras = Inverse[kets^T]}$$

$$\left\{\left\{\frac{1}{2}, \frac{1}{2}, 0, 0\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, 0, 0\right\}, \left\{0, 0, \frac{1}{2}, \frac{1}{2}\right\}, \left\{0, 0, -\frac{1}{2}, \frac{1}{2}\right\}\right\}$$

key step: now need to express V in the eigenbasis of H0

$$\mathbf{(Vp = bras.V.kets^T) \; // \; MatrixForm}$$

$$\begin{pmatrix} 0 & 0 & c+f & 0 \\ 0 & 0 & 0 & c-f \\ c+f & 0 & 0 & 0 \\ 0 & c-f & 0 & 0 \end{pmatrix}$$

note that Vp has no diagonal elements, so it has no effect to first order. Note also that Vp only has matrix elements between the two even and the two odd eigenstates.
We must therefore use second order perturbation theory. For the symmetric ground state, we get

$$\mathbf{e0s[[1]] + \frac{(c+f)^2}{e0s[[1]] - e0s[[3]]}}$$

$$-A1 + E1 + \frac{(c+f)^2}{-A1 + A2 + E1 - E2}$$

For the antisymmetric ground state, we get

$$\mathbf{e0s[\![2]\!] + \frac{(c - f)^2}{e0s[\![2]\!] - e0s[\![4]\!]}}$$

$$\mathrm{A1} + \mathrm{E1} + \frac{(\mathrm{c} - \mathrm{f})^2}{\mathrm{A1} - \mathrm{A2} + \mathrm{E1} - \mathrm{E2}}$$